

396 必背公式——高数篇

常用求导公式

$$C' = 0 \quad (C \text{ 为常数})$$

$$(\sec x)' = \sec x \tan x$$

$$(x^\alpha)' = \alpha x^{\alpha-1} \quad (\alpha \text{ 为常数})$$

$$(\csc x)' = -\csc x \cot x$$

$$(a^x)' = a^x \ln a \quad (a > 0, a \neq 1)$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\log_a x)' = \frac{1}{x \ln a} \quad (a > 0, a \neq 1)$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\tan x)' = \sec^2 x$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

$$(\cot x)' = -\csc^2 x$$

常用积分公式

以下公式中， α 与 a 均为常数，除括号内声明外， $a > 0$

$$\int k dx = kx + C \quad (k \text{ 为常数})$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \tan x dx = -\ln|\cos x| + C$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$$

$$\int \cot x dx = \ln|\sin x| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C$$

两个重要极限

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e = 2.718281828459045\dots$$

三角换元法

①含 $\sqrt{a^2 - x^2}$, 令 $x = a \sin t$, $dx = a \cos t dt$ ($a > 0$)

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C ;$$

②含 $\sqrt{x^2 + a^2}$, 令 $x = a \tan t$, $dx = a \sec^2 t dt$ ($a > 0$)

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 + a^2}) + C ;$$

③含 $\sqrt{x^2 - a^2}$, 令 $x = a \sec t$, $dx = a \sec t \tan t dt$ ($a > 0$)

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + C$$

④三角函数的有理式积分 (万能代换)

$$\text{令 } \tan \frac{x}{2} = t, \text{ 则 } \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2} dt$$

华里士公式 (点火公式)

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{1}{2} \times \frac{\pi}{2}, & \text{当 } n \text{ 为正偶数} \\ \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{2}{3} \times 1, & \text{当 } n \text{ 为大于1的正奇数} \end{cases}$$

常用无穷小替换

当 $x \rightarrow 0$ 时,

$$\sin x \sim x$$

$$\tan x \sim x$$

$$\arcsin x \sim x$$

$$\arctan x \sim x$$

$$1 - \cos x \sim \frac{1}{2}x^2$$

$$\ln(x+1) \sim x$$

$$(1+x)^a - 1 \sim ax$$

$$a^x - 1 \sim x \ln a \quad (a > 0, a \neq 1)$$

三角函数公式

➤ 和差化积公式

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha - \cos \beta = 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta - \alpha}{2}$$

➤ 和差角公式

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

$$\cot(\alpha \pm \beta) = \frac{1}{\tan(\alpha \pm \beta)}$$

➤ 半角公式

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}, \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}$$

$$\cot \frac{\alpha}{2} = \pm \sqrt{\frac{1+\cos \alpha}{1-\cos \alpha}} = \frac{1+\cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1-\cos \alpha}$$

➤ 倍角公式

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$$

$$\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

高阶导数公式——莱布尼兹公式

$$\begin{aligned} (uv)^{(n)} &= \sum_{k=0}^n C_n^k u^{(n-k)} v^{(k)} \\ &= u^{(n)} v + n u^{(n-1)} v' + \frac{n(n-1)}{2!} u^{(n-2)} v'' + \cdots + \frac{n(n-1) \cdots (n-k+1)}{k!} u^{(n-k)} v^{(k)} + \cdots + u v^{(n)} \end{aligned}$$

常见初等函数的 n 阶导数公式

$$(e^{ax})^{(n)} = a^n e^{ax}$$

$$(\sin ax)^{(n)} = a^n \sin\left(\frac{n\pi}{2} + ax\right)$$

$$(\cos ax)^{(n)} = a^n \cos\left(\frac{n\pi}{2} + ax\right)$$

$$(\ln(1+x))^{(n)} = \frac{(-1)^{n-1} (n-1)!}{(1+x)^n}$$

$$\left((1+x)^\alpha\right)^{(n)} = \alpha(\alpha-1) \cdots (\alpha-n+1)(1+x)^{\alpha-n},$$

若 α 为某一正整数 n ，则 $\left((1+x)^n\right)^{(n)} = n!$ ， $\left((1+x)^n\right)^{(n+j)} = 0$ ($j=1, 2, \cdots$)

中值定理与导数应用

拉格朗日中值定理： $f(b) - f(a) = f'(\xi)(b - a)$

柯西中值定理： $\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(\xi)}{F'(\xi)}$

当 $F(x) = x$ 时，柯西中值定理就是拉格朗日中值定理。

多元函数微分法及应用

全微分： $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$ $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$

全微分的近似计算： $\Delta z \approx dz = f_x(x, y)\Delta x + f_y(x, y)\Delta y$

多元复合函数的求导法：

$$z = f[u(t), v(t)] \quad \frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial t}$$

$$z = f[u(x, y), v(x, y)] \quad \frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

当 $u = u(x, y)$ ， $v = v(x, y)$ 时，

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

隐函数的求导公式：

$$\text{隐函数 } F(x, y) = 0, \quad \frac{dy}{dx} = -\frac{F_x}{F_y} \cdot \frac{d^2 y}{dx^2} = \frac{\partial}{\partial x^2} \left(-\frac{F_x}{F_y} \right) + \frac{\partial}{\partial y} \left(-\frac{F_x}{F_y} \right) \cdot \frac{dy}{dx}$$

$$\text{隐函数 } F(x, y, z) = 0, \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\text{隐函数方程组: } \begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases} \quad J = \frac{\partial(F, G)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{J} \cdot \frac{\partial(F, G)}{\partial(x, v)} \quad \frac{\partial v}{\partial x} = -\frac{1}{J} \cdot \frac{\partial(F, G)}{\partial(u, x)}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{J} \cdot \frac{\partial(F, G)}{\partial(y, v)} \quad \frac{\partial v}{\partial y} = -\frac{1}{J} \cdot \frac{\partial(F, G)}{\partial(u, y)}$$

多元函数的极值及其求法：

设 $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, 令 : $f_{xx}(x_0, y_0) = A$, $f_{xy}(x_0, y_0) = B$, $f_{yy}(x_0, y_0) = C$

$$\text{则: } \begin{cases} AC - B^2 > 0 \text{ 时, } \begin{cases} A < 0, (x_0, y_0) \text{ 为极大值} \\ A > 0, (x_0, y_0) \text{ 为极小值} \end{cases} \\ AC - B^2 < 0 \text{ 时, 无极值} \\ AC - B^2 = 0 \text{ 时, 不确定} \end{cases}$$

一阶线性微分方程的两个公式

齐次方程 : $y' + P(x)y = 0 \Rightarrow y = Ce^{-\int P(x)dx}$

非齐次方程 : $y' + P(x)y = Q(x) \Rightarrow y = e^{-\int P(x)dx} \left[\int Q(x)e^{\int P(x)dx} dx + C \right]$

旋转体体积

①曲线 $y = f(x)$ 与 $x = a$, $x = b$, x 轴围成的曲边梯形绕 x 轴旋转一周所成的旋转体体积：

$$V = \pi \int_a^b f^2(x) dx, a < b$$

②曲线 $y = y_2(x)$, $y = y_1(x)$, $x = a$, $x = b$ ($y_2(x) \geq y_1(x) \geq 0$) 围成的图形绕 x 轴旋转一周所成的旋转体体积：

$$V = \pi \int_a^b [y_2^2(x) - y_1^2(x)] dx, a < b$$

③曲线 $y = y_2(x)$, $y = y_1(x)$, $x = a$, $x = b$ ($b > a \geq 0$, $y_2(x) \geq y_1(x)$) 围成的图形绕 y 轴旋转一周所成的旋转体体积：

$$V = 2\pi \int_a^b x(y_2(x) - y_1(x)) dx$$

□级数部分会计算即可。