

引言:

数学150分, 23个题, 分四类问题: ①概念 ②计算 ③证明 ④应用

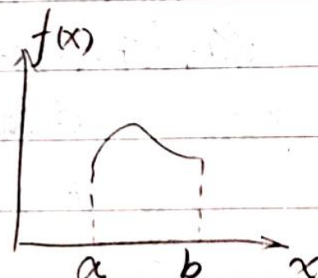
一. 概念型问题, 定义

1. 导数的定义 $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$ — 瞬时变化率[注] (1) 当 $\Delta x \rightarrow 0^+ \Rightarrow f'_+(x_0) = \lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ — 右导数当 $\Delta x \rightarrow 0^- \Rightarrow f'_-(x_0) = \lim_{\Delta x \rightarrow 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ — 左导数 Δ 数学规定: $f'(x)$ 存在 $\Leftrightarrow f'_+(x_0) = f'_-(x_0)$ (2) 变体形式 令 $x_0 + \Delta x = x \Rightarrow$ $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$, 等价(3) 泛化 $\Delta f'(x_0) \triangleq \lim_{t \rightarrow 0} \frac{f(x_0 + t) - f(x_0)}{t}$ (4) $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} \neq f'(x_0)$ 要求 x_0 处的变化率, 但无 x_0 出现坚持导数定义“一动一静”, 即 $f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ 例题: (1) 2018. 下列函数中在 $x=0$ 处不可导的A. $f(x) = |x| \sin |x|$ B. $f(x) = |x| \sin \sqrt{x}$ C. $f(x) = \cos |x|$ D. $f(x) = \cos \sqrt{|x|}$ (2) 设 $f(0) = 0$ 且 $\lim_{h \rightarrow 0} \frac{f(1 - \cosh h)}{h}$ 存在, 则 $f'(0)$ 存在?

2. 积分的定义 — 黎曼积分

(1). $S_{\text{曲边梯形}} = \int_a^b f(x) dx = \sum_{i=1}^n f(x_i) \Delta x$, $f(x) \Delta x$ — 小条面积[注] (1) $\int_a^b f(x) dx$ 是一个数 (面积)(2) ① 若 $f(x)$ 奇, $\Rightarrow \int_{-a}^a f(x) dx = 0$ ② 若 $f(x)$ 偶 $\Rightarrow \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ — “偶倍奇零”[例] 设 $M = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+x^2}{1+x^2} dx$, $N = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+x}{e^x} dx$, $K = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sqrt{\cos x}) dx$, 比大小[分析] $M = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \frac{x^2}{1+x^2} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx = \pi$ $N = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+x}{e^x} dx < \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx = M$, $e^x \geq 1+x$, $x=0$ 时相等 $K = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sqrt{\cos x}) dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx + 2 \int_0^{\frac{\pi}{2}} \sqrt{\cos x} dx > M$

(2). 精确积分定义

① 两个任意: 任意区间 $[a, b]$ 切分, 任意取高 $\Rightarrow \int_a^b f(x) dx$ 唯一则 $\exists$ 且唯一

② 考研中命题形式:

1° 将任意区间 n 等分 $\Rightarrow$ 每段长度 $(b-a)/n$ 2° 取在端点的高 $\Rightarrow S_{\text{梯}} = f(a + \frac{b-a}{n}) \times \frac{b-a}{n}, f(a + 2 \times \frac{b-a}{n}) \times \frac{b-a}{n}$ 故 $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(a + \frac{b-a}{n} i) \cdot \frac{b-a}{n}$ — “分割近似求和取极限”3° 取 $a=0, b=1 \Rightarrow \Delta \int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\frac{i}{n}) \cdot \frac{1}{n}$ [注] 凑微元三步曲: ① 提出 $\frac{1}{x}$ ② 凑出 $\frac{1}{x}$ ③ “ $\frac{1}{x}$ ”为 dx , “ $\frac{1}{x}$ ”为 $0 \sim 1$ 上的 x [例1] 求极限 $\lim_{n \rightarrow \infty} (\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n})$ “凑微元法则失效”[分析]: 原式 = $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n+i} \xrightarrow{\text{①}} \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{1+\frac{i}{n}} \times \frac{1}{n}$

$$= \int_0^1 \frac{1}{1+x} dx = \ln(1+x) \Big|_0^1 = \ln 2 \approx 0.69$$

[例2] 求极限 $\lim_{n \rightarrow \infty} (\frac{n+1}{n^2+1} + \frac{n+2}{n^2+4} + \frac{n+3}{n^2+9} + \dots + \frac{n+n}{n^2+n^2})$ [分析]: 原式 = $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n+i}{n^2+i^2} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1+\frac{i}{n}}{1+(\frac{i}{n})^2} \times \frac{1}{n}$

$$= \int_0^1 \frac{1+x}{1+x^2} dx = \arctan x \Big|_0^1 + \frac{1}{2} \ln(1+x^2) \Big|_0^1 = \frac{\pi}{4} + \frac{1}{2} \ln 2$$

二. 计算型问题

工. 三大基础: 极限计算 导数计算 积分计算 公式

 Δ 泰勒公式(计算): 任何可导函数都是 x^n 组成, “ $f(x) = \sum n x^n$ ” $\Rightarrow$ 统一形式美常见展开式: $x \rightarrow 0$ ① $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + o(x^3)$

$$\text{② } \arcsin x = x + \frac{x^3}{6} + o(x^3) \quad \text{③ } \tan x = x + \frac{x^3}{3} + o(x^3)$$

$$\text{④ } \arctan x = x - \frac{x^3}{3} + o(x^3)$$

$$x - \sin x = x - (x - \frac{1}{6}x^3 + o(x^3)) = \frac{1}{6}x^3 + o(x^3) \sim \frac{1}{6}x^3$$

$$[Th] A = B + \alpha, \alpha \text{ 为高阶无穷小} \Rightarrow A \sim B \Leftrightarrow \lim_{x \rightarrow 0} \frac{A}{B} = 1$$

$$\Delta t - \sin t = \frac{1}{6}t^3 (t \rightarrow 0), \text{ 广义化}$$

$$[例1] 10' \lim_{x \rightarrow 0} \frac{\sin x - \sin(\sin x)}{x^4}$$

$$[分析] \text{ 原式} = \lim_{x \rightarrow 0} \frac{\frac{1}{6}(\sin x)^3 \times \sin x}{x^4} = \frac{1}{6}$$

$$[例2] \text{ 若 } \lim_{x \rightarrow 0} \frac{x - \sin ax}{bx^3} = 1, \text{ 求 } a, b$$

$$[分析] \sin ax = ax - \frac{1}{6}a^3x^3, x - \sin ax = (1-a)x + \frac{1}{6}a^3x^3 + o(x^3)$$

$$\text{故 } a=1, b=\frac{1}{6}a^3=\frac{1}{6}$$

[例3] 若 $\lim_{x \rightarrow 0} \frac{x - \arctan kx}{cx^3} = 1$, 求 k, c

[分析] $\arctan kx = kx - \frac{1}{3}k^3x^3 + o(x^3)$

故 $k=1, c = +\frac{1}{3}k^3 = +\frac{1}{3}$

[例4] $\lim_{x \rightarrow 0} \frac{\arcsin x - \arctan x}{\sin x - \tan x} = -1$

[分析] $(\arcsin x - \arctan x) \sim x + \frac{1}{6}x^3 - (x - \frac{1}{3}x^3) + o(x^3) \sim \frac{1}{2}x^3$

$(\sin x - \tan x) = x - \frac{1}{6}x^3 - (x + \frac{1}{3}x^3) + o(x^3) \sim -\frac{1}{2}x^3$

三. 证明型问题(逻辑推理): ①定理+②思路 训练大量

△ 费马Th: (1) 若 $f(x)$ 在 x_0 处可导且极值, 则 $f'(x_0)=0$

罗尔Th: 设 $f(x)$ 满足以下三个条件: (1) $[a, b]$ 上连续 (2) (a, b) 内可导

(3) $f(a)=f(b) \Rightarrow \exists \xi \in (a, b)$, 使 $f'(\xi)=0$. “(3) $\Rightarrow$ 有极值”

[注] 罗尔原话: 若 $f(x)=0$ 有两个根 $\Rightarrow f(x)=0$ 至少有一个根.

逆否: $f(x)=0$ 没有根, 至多0个 $\Rightarrow f(x)=0$ 至多1个根

数学归纳: 若 $f(x)=0$ 有 k 个根 $\Rightarrow f(x)=0$ 有 $(k+1)$ 个根

△ 罗尔推论Th: 若 $f^{(n)}(x)=0$ 至多 k 个根 $\Rightarrow f^{(n-1)}(x)=0$ 至多 $(k+1)$ 个根.

[例] 证明 $\ln x - e^x + 2\sqrt{e} = 0$ 有且仅有两个根.

[分析] ①有: 存在性零点定理. 令 $f(x) = \ln x - e^x + 2\sqrt{e}$, $(0, +\infty)$

零点定理: 若 $f(x)$ 在 $[a, b]$ 上连续且 $f(a) \cdot f(b) < 0 \Rightarrow \exists \xi \in (a, b)$, 使 $f(\xi)=0$

[注] 若 a, b 取不到, 如 (a, b) , 则求 $\lim_{x \rightarrow a^+}, \lim_{x \rightarrow b^-}$.

$$\lim_{x \rightarrow 0^+} (\ln x - e^x + 2\sqrt{e}) = -\infty < 0$$

$$\lim_{x \rightarrow +\infty} (\ln x - e^x + 2\sqrt{e}) = -\infty < 0$$

$$f(1) = \ln 1 - e + 2\sqrt{e} \approx 2.8 - 2.7 > 0$$

$\Rightarrow (0, 1), \exists \xi_1$ 且 $f(\xi_1)=0$
 $(1, +\infty), \exists \xi_2$ 且 $f(\xi_2)=0$
 即至少两根

②证“仅有”(唯一性): 法一“单调性”若 $f(x) > 0 \Rightarrow f(x) \uparrow$. 分两个区间, $x_0 \in (0, 1)$

对此 $f'(x) = \frac{1}{x} - e^x$, 令 $f'(x)=0$, 不可得故不可用.

法二: $f^{(n)}(x)=0$ 至多 k 个根 $\Rightarrow f^{(n-1)}(x)=0$ 至多 $(k+1)$ 个根

故 $f'(x) = \frac{1}{x} - e^x$, $f''(x) = -\frac{1}{x^2} - e^x < 0 \neq 0$

$f''(x)$ 至多0个根 $\Rightarrow f'(x)$ 至多1个根 $\Rightarrow f(x)$ 至多2个根, 命题可证.

四. 应用型问题

把数学知识用到解决数学问题上, 其它数学问题.

实际背景

纯数学问题

1. 泰勒公式: 若 $f(x)$ n 阶可导 $\Rightarrow f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(\xi)}{n!}(x-x_0)^n$

$\uparrow$ 三项 $\leftarrow$ 拉格朗日余项 (推理)

故 $\sin x \stackrel{x_0=0}{=} \sin x_0 + \cos x_0(x-0) - \sin 0 \cdot \frac{x^2}{2!} - \cos 0 \cdot \frac{x^3}{3!} + \cos \xi \cdot \frac{x^5}{5!}$

常考: $f(x)$ 二阶可导 $\Rightarrow f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2!}(x-x_0)^2$ ξ 在 x 与 x_0 之间.

[例] 设 $f(x)$ 在 $[0, 1]$ 二阶可导且 $\int_0^1 f(x) dx = 0$, 证明当 $f'(x) > 0$ 时, 必有 $f(\frac{1}{2}) < 0$.

[分析] 见到 f, f' $\Rightarrow$ 泰勒公式; 见到 f, f' $\Rightarrow$ 拉格朗日

故 $f(x) = f(\frac{1}{2}) + f'(\frac{1}{2})(x-\frac{1}{2}) + \frac{f''(\xi)}{2!}(x-\frac{1}{2})^2$

由 $f'(x) > 0 \Rightarrow f'(\xi) > 0 \Rightarrow \frac{f''(\xi)}{2!}(x-\frac{1}{2})^2 > 0$

$f(x) > f(\frac{1}{2}) + f'(\frac{1}{2})(x-\frac{1}{2}) \Rightarrow \int_0^1 f(x) dx = 0 > \int_0^1 [f(\frac{1}{2}) + f'(\frac{1}{2})(x-\frac{1}{2})] dx$

$= f(\frac{1}{2}) + \int_0^1 f'(\frac{1}{2})(x-\frac{1}{2}) dx = f(\frac{1}{2}) + \int_{-\frac{1}{2}}^{\frac{1}{2}} f'(\frac{1}{2})x dx = f(\frac{1}{2})$

极限

核心考点: (1) 定义-4' (2) 性质-4' Δ (3) 计算-10+4' (数列) (4) 应用-4' (间断)

一、极限定义

1. 函数极限 $\lim_{x \rightarrow x_0} f(x) = A \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, 0 < |x - x_0| < \delta, |f(x) - A| < \varepsilon$

$f(x)$ 有四种趋势: $f(x) \rightarrow A, f(x) \rightarrow \infty, f(x) \rightarrow -\infty, f(x) \rightarrow +\infty$

x 有六种趋势: $x \rightarrow x_0, x \rightarrow x_0^-, x \rightarrow x_0^+, x \rightarrow \infty, x \rightarrow +\infty, x \rightarrow -\infty$

局部: $x \rightarrow x_0 \Rightarrow \exists \delta > 0, 0 < |x - x_0| < \delta; x \rightarrow x_0^- \Rightarrow \exists \delta > 0, x_0 - \delta < x < x_0; x \rightarrow x_0^+ \Rightarrow \exists \delta > 0, 0 < x - x_0 < \delta$

无穷: $x \rightarrow \infty \Rightarrow \exists X > 0, |x| > X; x \rightarrow +\infty \Rightarrow \exists X > 0, x > X; x \rightarrow -\infty \Rightarrow \exists X > 0, -x > X$

$f(x)$: $f(x) \rightarrow \infty \Rightarrow \exists M > 0, |f(x)| > M; f(x) \rightarrow -\infty \Rightarrow \exists M > 0, f(x) < -M$; 并组合即得定义

例: $\lim_{x \rightarrow \infty} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists X > 0, \text{当 } |x| > X, f(x) > M$

2. 数列极限 $\{a_n\}, n \rightarrow \infty$ 指 $n \rightarrow +\infty$

$\lim_{n \rightarrow \infty} a_n = A \Leftrightarrow \forall \varepsilon > 0, \exists N > 0, n > N, 0 < |a_n - A| < \varepsilon$

[例] 以下三个命题

(1) 若 $\{u_n\}$ 收敛于 A , 则其任意子数列 $\{u_{n_i}\}$ 必收敛于 A

(2) 若单调数列 $\{x_n\}$ 的某子数列 $\{x_{n_i}\}$ 收敛于 A , 则该数列必收敛于 A .

(3) 若 $\{x_{2n}\}$ 与 $\{x_{2n+1}\}$ 均收敛于 A .

[分析] (1) $\lim_{n \rightarrow \infty} u_n = A \Leftrightarrow \forall \varepsilon > 0, \exists N > 0, n > N, 0 < |u_n - A| < \varepsilon$

故当 $n_i > N$ 时 (解之即可), $0 < |u_{n_i} - A| < \varepsilon$, 命题得证.

Δ (2) 设该数列单调, 即 $x_1 \leq x_2 \leq \dots \leq x_n \leq \dots$,

$\lim_{n \rightarrow \infty} u_{n_i} = A \Leftrightarrow \forall \varepsilon > 0, \exists N > 0, n_i > N, 0 < |u_{n_i} - A| < \varepsilon$

且必定有在 $n_i \leq n \leq n_{i+1}$ 故 $u_{n_i} \leq u_n \leq u_{n_{i+1}}$ (单调子数列)

$\Rightarrow \varepsilon < u_{n_i} - A \leq x_n - A \leq u_{n_{i+1}} - A < \varepsilon \Rightarrow 0 < |u_n - A| < \varepsilon$, 易知收敛

(3) 当 $2n > N_1$ 时 $|x_{2n} - A| < \varepsilon$; 当 $2n+1 > N_2$ 时 $|x_{2n+1} - A| < \varepsilon$

当 $N = \max\{N_1, N_2\}$, $n > N$ 时 $|x_n - A| < \varepsilon$, 即 $\lim_{n \rightarrow \infty} x_n = A$

二、极限三大性质

1. 唯一性 若 $\lim_{x \rightarrow x_0} f(x) = A$, 则 A 唯一。取 $\varepsilon = (A \pm B)/2$

[例] 设 a 为常数, $I = \lim_{x \rightarrow 0} (\frac{e^x - \pi}{e^{\frac{1}{x}} + 1} + a \arctan \frac{1}{x})$ 存在, 求 a, I (左右有别)

[分析] 从 $x \rightarrow 0^+$, $x \rightarrow 0^-$ 进行分析

$$(1) \lim_{x \rightarrow 0^+} \frac{e^x - \pi}{e^x + 1} = \lim_{x \rightarrow +\infty} \frac{e^x}{e^x} = 0; \lim_{x \rightarrow 0^+} a \arctan \frac{1}{x} = \frac{a\pi}{2}$$

$$(2) \lim_{x \rightarrow 0^-} \frac{e^x - \pi}{e^x + 1} = \lim_{x \rightarrow 0^-} \frac{-\pi}{1} = -\pi, \lim_{x \rightarrow 0^-} -a \arctan \frac{1}{x} = -\frac{a\pi}{2}$$

由唯一性知 $\frac{a\pi}{2} = -\frac{a\pi}{2} - \pi \Rightarrow a=1, I = -\frac{\pi}{2}$

$$|a \pm b| \leq |a| + |b|$$

$\Delta 2$. 局部有界性 若 $\lim_{x \rightarrow x_0} f(x) = A \Rightarrow \exists M > 0, \delta > 0$ 当 $0 < |x - x_0| < \delta$ 有 $|f(x)| < M$

恒等变换

证明: $\lim_{x \rightarrow x_0} f(x) = A \Rightarrow \forall \varepsilon > 0 \exists \delta > 0, 0 < |x - x_0| < \delta, 0 < |f(x) - A| < \varepsilon$

$$|f(x)| = |f(x) - A + A| \leq |f(x) - A| + |A|$$

取任意 $\varepsilon > 0 \Rightarrow \varepsilon > |f(x) - A| > |f(x)| - |A| \Rightarrow |f(x)| < |A| + \varepsilon$

令 $\varepsilon = \frac{|A|}{2}$, 即 $|f(x)| < \frac{3}{2}|A|$, 令 $M = \frac{3}{2}|A|$ (一个数)

得当 $0 < |x - x_0| < \delta$, $|f(x)| < M$, 有界性得证。

Δ [例1] $f(x) = \frac{|x| \sin(x-2)}{x(x-1)(x-2)}$ 在 () 内有界

A. (-1, 0) B. (0, 1) C. (1, 2) D. (2, 3)

[分析] 讨论 $f(x)$ 在指定 I 的有界性, 方法总结如下:

(1) 理论法: 若 I 为 $[a, b]$ 闭区间, 用“连续 $f(x)$ 在 $[a, b]$ 上必有界”

(2) 计算: 若 I 为 (a, b) 开区间, 用“三段论”:

① $\lim_{x \rightarrow a^+} f(x) \exists \Rightarrow x \in (a, a+\delta), |f(x)| < M_1$,

② $\lim_{x \rightarrow b^-} f(x) \exists \Rightarrow x \in (b-\delta, b), |f(x)| < M_2$,

③ 若 $f(x)$ 在 (a, b) 内连续, 则 $f(x)$ 有界, (夹逼, 必成立)

该 $f(x)$ 为初等函数, 在其定义区间内连续。

$\lim_{x \rightarrow -1+0} f(x) = \sin 3 / 6 \neq \infty$; $\lim_{x \rightarrow 0-0} f(x) = \frac{\sin 2}{2} \neq \infty$; $\lim_{x \rightarrow 0+0} f(x) = -\frac{\sin 2}{2} \neq \infty$

而 $\lim_{x \rightarrow 1^-} f(x)$, $\lim_{x \rightarrow 1^+} f(x)$, $\lim_{x \rightarrow 2^+} f(x)$, $\lim_{x \rightarrow 2^-} f(x)$ 而面均不存在, 故 $(-1, 0)$ 上有界。

$\Delta 3$. 局部保号性 $\lim_{x \rightarrow x_0} f(x) = A > 0 \Rightarrow x \rightarrow x_0$ 时, $f(x) > 0$

证明: $\lim_{x \rightarrow x_0} f(x) = A \Rightarrow \forall \varepsilon > 0, \exists \delta > 0, 0 < |x - x_0| < \delta, |f(x) - A| < \varepsilon$

$\Rightarrow A - \varepsilon < f(x) < A + \varepsilon$ 令 $\varepsilon = \frac{A}{2} \Rightarrow f(x) > \frac{A}{2} > 0$, 命题得证。

[注] $\lim_{x \rightarrow x_0} f(x) = A \Rightarrow A - \varepsilon < f(x) < A + \varepsilon \Rightarrow \frac{A}{2} < f(x) < \frac{3A}{2}$

“极限无限小, 区间较宏观” $-A < f(x) < 3A$

仅仅是一个陈述
非取值范围。

[例] $\lim_{x \rightarrow 0} f(x) = f(0)$ 且 $\lim_{x \rightarrow 0} \frac{f(x)}{1 - \cos x} = 2$, 则 $x=0$ 是 ()

[分析] 讨论极值必须有双侧邻域, 双侧定义

$$f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{f(x)}{1 - \cos x} \cdot (1 - \cos x) = \lim_{x \rightarrow 0} 2(1 - \cos x) = 0 \quad \text{"恒等变换"}$$

$$\because \lim_{x \rightarrow 0} f(x) / (1 - \cos x) < 0 \Rightarrow \frac{f(x)}{1 - \cos x} < 0 \quad (x \rightarrow x_0), \cos x < 1 \Rightarrow f(x) < 0 \quad (x \rightarrow x_0)$$

故为极大值点。 "脱帽法": 在 $f(x)$ 性态中的研究

三. 极限的计算

1. 函数极限计算: 常规

(1) 七种未定型: $0 \cdot \infty, \frac{0}{0}, \frac{\infty}{\infty}; \infty - \infty; 1^\infty, \infty^0, 0^\infty$. (均为极限 $1, 0$)

(2) 计算工具 ① 洛必达法则

a) 若 $\lim_{x \rightarrow x_0} f(x) = 0, \lim_{x \rightarrow x_0} g(x) = 0$ 且 $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ 存在, 则 $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ (前提)

[注] ① $\lim_{x \rightarrow 0} \frac{\sin x}{x} \stackrel{0}{=} \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$ E. ② 满足前提, 适用

$\lim_{x \rightarrow \infty} \frac{\sin x + x}{x} \stackrel{\infty}{=} \lim_{x \rightarrow \infty} \frac{1 + \cos x}{1}$, 不存在, 不满足前提, 不适用洛必达

故洛必达法则 "能不能用, 用了再说", 否则, 若不存在, 换其它方法

② 常用等价无穷小: $(1+x)^a - 1 \sim ax \quad (x \rightarrow 0)$. [注] $(1 + \sin x)^{\frac{1}{3}} - 1 \sim \frac{1}{3} \sin x$, 广义化

[例] 第组 $(0 \cdot \infty, \frac{0}{0}, \frac{\infty}{\infty})$.

$$[例1] \lim_{x \rightarrow 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{(x\sqrt{1+\sin^2 x} - x)} \quad (\frac{0}{0})$$

[分析] 化简先行: 见更号差用有理化 $(\sqrt{a} - \sqrt{b})$

$$\begin{aligned} \text{原式} &= \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x(\sqrt{1+\sin^2 x} - 1)(\sqrt{1+\tan^2 x} + \sqrt{1+\sin^2 x})} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^3/2x \cdot \frac{1}{2} \cdot \sin^2 x}{x^3} = \frac{1}{4} \quad \left(\lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right) \quad \text{提公因式} \\ &= \lim_{x \rightarrow 0} \frac{1}{\cos x} - 1/x^2 = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2 \cos x} = \frac{1}{2} \quad \Delta \uparrow \tan x (1 - \cos x) \end{aligned}$$

△ [注] 常规化简方法: 1) 恒等变形 (有理化, 提公因式, $+$, $\times$ 等)

2) 等价无穷小替换, 勿忘 x 的广义化. 3) 及时提出极限不为 0 的因式 ✓

$$[例2] \lim_{x \rightarrow 0} \frac{e^{x^2} - e^{2-2\cos x}}{x^4} \quad (\frac{0}{0})$$

$$[分析1] \text{原式} = \lim_{x \rightarrow 0} \frac{e^{x^2-2+2\cos x} (e^{x^2-2+2\cos x} - 1)}{x^4} = \lim_{x \rightarrow 0} e^{x^2-2+2\cos x} \cdot \frac{x^2+2\cos x-2}{x^4}$$

$$= 1 \cdot \lim_{x \rightarrow 0} \left[x^2 + 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) - 2 \right] / x^4 = \frac{1}{12}$$

第 二 组

$$[例3] \lim_{x \rightarrow 1^-} \ln x \cdot \ln(1-x) \quad (0 \cdot \infty)$$

[分析] "0·∞" 两种化为 $\frac{\infty}{\infty}$ 或 $\frac{0}{0}$

正三角: 分子简单

$$\lim_{x \rightarrow 0^+} x \ln x \xrightarrow{0 \cdot \infty} \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \xrightarrow{\frac{-\infty}{\infty}} \lim_{x \rightarrow 0^+} -\frac{1}{x} \cdot \frac{1}{x^2} = -x \ln^2 x, \text{ 变复杂 分解复杂 } \langle X \rangle!$$

$$\lim_{x \rightarrow 0^+} x \ln x \xrightarrow{0 \cdot \infty} \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \xrightarrow{\frac{-\infty}{\infty}} \lim_{x \rightarrow 0^+} \frac{1}{x^2} = -\lim_{x \rightarrow 0^+} x = 0 \text{ (倒三角)}$$

[小结] "设置分母有原则, 简单因式下放", 形成倒三角.

简单: x^d, e^{Bx} 复杂: $\ln x, \arcsin x, \arctan x$ etc.

$$\text{原式} = \lim_{x \rightarrow 1^-} \ln(x+1) \ln(1-x) = \lim_{x \rightarrow 1^-} (x-1) \ln(1-x) = \lim_{x \rightarrow 1^-} \frac{\ln(1-x)}{\frac{1}{x-1}} \text{ (倒三角)}$$

$$\xrightarrow{\frac{0}{\infty}} \lim_{x \rightarrow 1^-} -\frac{1}{1-x} / -\frac{1}{(x-1)^2} = \lim_{x \rightarrow 1^-} 1-x = 0$$

第二组 ($\infty - \infty$)

① 有分母, 则通分.

$$[例] \lim_{x \rightarrow 0} \left(\frac{e^x + xe^x}{e^x - 1} - \frac{1}{x} \right) (\infty - \infty)$$

$$[分析] \text{原式} = \lim_{x \rightarrow 0} \frac{1}{x(e^x - 1)} (xe^x + xe^x - e^x + 1) = \lim_{x \rightarrow 0} \frac{1}{x^2} (xe^x + xe^x - e^x + 1)$$

$$= \lim_{x \rightarrow 0} e^x + \lim_{x \rightarrow 0} \frac{xe^x - e^x + 1}{x^2} = 1 + \lim_{x \rightarrow 0} \frac{xe^x + e^x e^x}{2x} = \frac{3}{2}$$

② 没有分母, 创造分母 - 值代换令 $x = \frac{1}{t}$

$$[例] \lim_{x \rightarrow +\infty} [x(e^{\frac{1}{x}} - 1) - x] (\infty - \infty) = \frac{1}{2}$$

$$[分析] \text{原式} = \lim_{t \rightarrow 0^+} \frac{e^t - 1}{t^2} - \frac{1}{t} = \lim_{t \rightarrow 0^+} \frac{e^t - 1 - t}{t^2} = \lim_{t \rightarrow 0^+} \frac{1+t+\frac{t^2}{2}-1-t+o(t)}{t^2} = \frac{1}{2}$$

第三组 ($\infty^0, 0^0, 1^\infty$)

$$u(x)v(x) = e^{v(x) \ln u(x)}, \text{ (幂指函数, 初等函数)}$$

$$[例1] \lim_{x \rightarrow +\infty} (x + \sqrt{1+x^2})^{\frac{1}{x}} (\infty^0)$$

$$[分析] \text{原式} = e^{\lim_{x \rightarrow +\infty} \frac{\ln(x + \sqrt{1+x^2})}{x}} \xrightarrow{\frac{\infty}{\infty}} \text{Exp} \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{1+x^2}} = 1$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \ln|x + \sqrt{1+x^2}|$$

$$[例2] \lim_{x \rightarrow \frac{\pi}{4}} (\tan x)^{\frac{\cos x - \sin x}{\cos x + \sin x}} (1^\infty)$$

$$[分析] \text{原式} = \lim_{x \rightarrow \frac{\pi}{4}} e^{\frac{\ln \tan x}{\frac{\cos x - \sin x}{\cos x + \sin x}}} = \lim_{x \rightarrow \frac{\pi}{4}} e^{\frac{\tan x - 1}{\cos x - \sin x}} \xrightarrow{\frac{0}{0}} \lim_{x \rightarrow \frac{\pi}{4}} e^{-\frac{1}{\cos x}} = e^{-\sqrt{2}}$$

$$[小结] 1^\infty \Rightarrow u^v = e^{v \ln u} = e^{v(u-1)}$$

2. 泰勒公式 任何可导函数 $f(x) = \sum a_n x^n$

$$x \rightarrow 0 \quad \cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + o(x^4); \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4); \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + o(x^3) \quad (|x| < 1)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + o(x^3)$$

例1] $\lim_{x \rightarrow 0} \frac{\arcsin x - \arctan x}{x^3}$

分析] 见到 $\frac{0}{0}$, 用“上下同阶”原则——若分母()为 x^k , 则分子展开至 x^k . 若分子()是 x^k , 分母展开为 x^k .

$$I = \lim_{x \rightarrow 0} \frac{x + \frac{x^3}{6} - (x - \frac{x^3}{3}) + o(x^3)}{x^3} = 1/2$$

例2] 当 $x \rightarrow 0$ 时, $\cos x - e^{-\frac{x^2}{2}}$ 与 Cx^k 为阶无穷小, 求 C, k .

分析] 见到“A-B”型, 用幂次最低原则——即将A, B分别展开

系 $\rightarrow$ 至次数不相等的 x 的最低次幂为止.

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2} + \frac{1}{4!}x^4 + o(x^4) \\ e^{-\frac{x^2}{2}} &= 1 - \frac{x^2}{2} + \frac{1}{2!}(\frac{x^2}{2})^2 + o(x^4) \end{aligned} \quad \begin{aligned} \cos x - e^{-\frac{x^2}{2}} &= -\frac{1}{12}x^4 + o(x^4) \sim Cx^k \\ \therefore C &= -\frac{1}{12} \quad k=4 \end{aligned}$$

[注] $x \rightarrow 0$, $\sin x \sim x$, $1 - \cos x \sim \frac{x^2}{2}$, $x - \sin x \sim \frac{x^3}{6}$, $x + \sin x \sim 2x$

$$\begin{aligned} \text{如 } \lim_{x \rightarrow 0} (\frac{1}{\sin x} - \frac{1}{x}) &\stackrel{\infty-\infty}{=} \lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^2 \sin x} = \lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^4} \\ &= \lim_{x \rightarrow 0} \frac{(x + \sin x)(x - \sin x)}{x^4} = \lim_{x \rightarrow 0} \frac{2x \cdot \frac{x^3}{6}}{x^4} = \frac{1}{3} \quad \text{故 } x^2 - \sin^2 x \sim \frac{x^4}{3} \end{aligned}$$

例3] 设 $f(x)$ 在 $x=0$ 的某领域内有定义, 且 $\lim_{x \rightarrow 0} \frac{f(x)\tan x - \sin 4x}{x^3} = 0$,

计算 $\lim_{x \rightarrow 0} \frac{f(x)-4}{x^2}$.

分析] $f(x)$ 抽象函数 $\Rightarrow$ 干掉 $f(x)$! $\Leftrightarrow$ 恒等变换

$$\textcircled{2} x \tan x \Rightarrow \lim_{x \rightarrow 0} \frac{f(x)\tan x - 4\tan x}{x^2 \tan x} = I = \lim_{x \rightarrow 0} \frac{f(x)\tan x - 4\tan x}{x^3}$$

$$I - \textcircled{1} \Rightarrow I = \lim_{x \rightarrow 0} \frac{\sin 4x - 4\tan x}{x^3} \quad (\text{“A-B”型})$$

$$\textcircled{1} \sin 4x = 4x - \frac{1}{3!}(4x)^3 + o(x^3) \quad \sin 4x - 4\tan x = -12x^3$$

$$4\tan x = 4x + \frac{4}{3}x^3 + o(x^3) \quad \text{故 } I = -12$$

$$\Delta \textcircled{2} \lim_{x \rightarrow 0} \sin 4x - 4\tan x = \lim_{x \rightarrow 0} \sin 4x - 4x + 4x - 4\tan x$$

$$= \lim_{x \rightarrow 0} (\sin 4x - 4x) + \lim_{x \rightarrow 0} 4x - 4\tan x = \lim_{x \rightarrow 0} -\frac{1}{6}(4x)^3 + 4 \lim_{x \rightarrow 0} \frac{x^3}{3} \sim -12x^3$$

四. 数列极限计算

1. 若 $\{x_n\}$ 易于极限化, 转化为函数极限计算.

依据: 若 $\lim_{x \rightarrow \infty} f(x) = A$, 则 $\lim_{n \rightarrow \infty} f(n) = A$. “海涅定理” “归结定理”

例] $\lim_{n \rightarrow \infty} (n \cdot \tan \frac{1}{n})^{n^2}$

分析] $n \rightarrow \infty \Rightarrow \lim_{x \rightarrow \infty} (x \cdot \tan \frac{1}{x})^{x^2} = A \Rightarrow \lim_{n \rightarrow \infty} (n \cdot \tan \frac{1}{n})^{n^2} = A$

故 $I = \lim_{x \rightarrow +\infty} (x \cdot \tan \frac{1}{x})^x (1^\infty) = \lim_{x \rightarrow +\infty} e^{x^2(x \cdot \tan \frac{1}{x} - 1)}$

令 $x = \frac{1}{t}$ $\lim_{t \rightarrow 0^+} \frac{\tan t - t}{t^3} = \lim_{t \rightarrow 0^+} \frac{\frac{1}{3}t^3}{t^3} = \frac{1}{3}$

根据洛必达定理, 原式 $= e^{\frac{1}{3}}$

2. 若 $\{x_n\}$ 不易于连续化, 用夹逼准则 (或定积分法)

① $y_n \leq x_n \leq z_n$ ② $n \rightarrow \infty$ $y_n \rightarrow A, z_n \rightarrow A$ ③ $n \rightarrow +\infty, x_n \rightarrow A$

例1] 求 $\lim_{n \rightarrow \infty} (\frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \dots + \frac{n}{n^2+n+n})$

[分析] 原式 $= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i}{n^2+n+i}$, “分子分母同变化, 只动分母不动分子”

$\sum_{i=1}^n \frac{i}{n^2+2n} < \sum_{i=1}^n \frac{i}{n^2+n+i} < \sum_{i=1}^n \frac{i}{n^2+n+1} = \frac{n(n+1)}{2} / (n^2+n+1)$

$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i}{2(n^2+2n)} = \frac{1}{2}; \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i}{n^2+n+1} = \lim_{n \rightarrow \infty} \frac{n^2+n}{2(n^2+n+1)} = \frac{1}{2}$

$\therefore \frac{1}{2} < \text{原式} < \frac{1}{2} \Rightarrow \text{原式} = \frac{1}{2}$ (夹逼准则)

例2] $\lim_{n \rightarrow \infty} \sqrt[n]{n \arctan n}$

[分析] 抓隐藏条件 $\Rightarrow$ 函数天生的“有界性” $\Rightarrow \frac{\pi}{4} < \arctan n < \frac{\pi}{2}$

$\frac{\pi}{4} \cdot n < \arctan n \cdot n < n \cdot \frac{\pi}{2}$

$\therefore \lim_{n \rightarrow \infty} (\frac{\pi}{4} \cdot n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} e^{\frac{\ln n}{n}} = 1 = \lim_{n \rightarrow \infty} (n \cdot \frac{\pi}{2})^{\frac{1}{n}}$

$\therefore$ 由夹逼准则得 $\lim_{n \rightarrow \infty} \sqrt[n]{n \arctan n} = 1$

3. 若 $\{x_n\}$ 由递推式 $x_n = f(x_{n-1})$ 给出, 用“单调有界准则”

$\triangle \triangle$ 给出 $\{x_n\}$, 若 $\{x_n\}$ 单调且有上界 $\Rightarrow \lim_{n \rightarrow \infty} x_n$ 存在 $\Leftrightarrow \{x_n\}$ 收敛

[例] 设 $x_1 = 1, x_{n+1} = 1 + \frac{x_n}{1+x_n} (n=1, 2, 3, \dots)$, 求 $\lim_{n \rightarrow \infty} x_n$.

[分析] “先证 $\exists$, 再求极限” $x_1 = 1, x_2 = \frac{3}{2} > x_1 > 0$. ① 验证 $n=1$ 成立

(第1归纳) ② 设 $n-1$ 成立, 设 $x_n > x_{n-1} > 0$

法步骤) ③ 证 n 成立. 则 $x_{n+1} - x_n = (1 + \frac{x_n}{1+x_n}) - (1 + \frac{x_{n-1}}{1+x_{n-1}}) = \frac{x_n - x_{n-1}}{(1+x_n)(1+x_{n-1})}$

由②的假设得 $x_{n+1} - x_n > 0$ 即 $x_{n+1} > x_n > 0$, 故

故由数学归纳法知 $\{x_n\}$ 是单调递增.

又 $x_{n+1} = 1 + \frac{x_n}{1+x_n} = 2 - \frac{1}{1+x_n} < 2 \therefore \{x_n\}$ 有上界 2.

$\Rightarrow \lim_{n \rightarrow \infty} x_n$ 存在 $A \Rightarrow n \rightarrow \infty$ 时, $A = 1 + \frac{A}{1+A}$

$\Rightarrow A^2 - A - 1 = 0 \Rightarrow A = \frac{1+\sqrt{5}}{2} (\because x_n > 0 \Rightarrow \lim_{n \rightarrow \infty} x_n = \frac{1+\sqrt{5}}{2})$

五. 极限的应用—连续与间断

1. 常识: 研究两类特殊点: (1) 分段函数分段点 (可能间断), 在不同区间表达式不同.
(2) 无定义点 (必然间断)

2. 连续的定义: $\lim_{x \rightarrow x_0} f(x) = f(x_0) \Rightarrow f(x)$ 在 x_0 处连续

[注] $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} f(x) = f(x_0)$, 才是连续.

3. 间断的定义: 设 $f(x)$ 在 $x = x_0$ 点的某去心邻域有定义 (双侧定义) ← 前提

① $\lim_{x \rightarrow x_0^-} f(x)$ ② $\lim_{x \rightarrow x_0^+} f(x)$ ③ $f(x_0)$

(1) 第一类间断点 ①, ② 均存在, 则

③ $\neq$ ① = ②, 为可去间断点; ① $\neq$ ②, 为跳跃间断点.

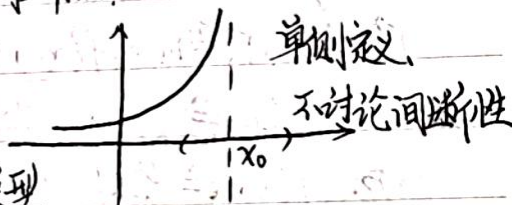
(2) 第二类间断点, ①, ② 至少一个不存在, 则 (目前 ①, ② 都不存在)

若不存在 = $\infty \Rightarrow$ 无穷间断点; 若不存在 = 振荡 $\Rightarrow$ 振荡间断点.

[注] 1. 定单侧定义不讨论间断性

2. 若左右一侧振荡间断, 一边是无穷间断, 则分别讨论

[例] 求 $f(x) = \begin{cases} \frac{x(\pi+2x)}{2\cos x}, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$ 的间断点, 并判断类型.



[分析] (1) 分段点 $x=0$, $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{x(\pi+2x)}{2\cos x} = 0$

$\lim_{x \rightarrow 0^+} \sin \frac{1}{x} = -\sin 1$, $f(0) = 0$.

故 $x=0$ 为跳跃间断点.

(2) 无定义点: $x > 0$ 时则 $x \neq 0$ 为无定义点, $\lim_{x \rightarrow 0^+} \sin \frac{1}{x} = \sin +\infty$ 振荡不存在

$x < 0$ 时则 $x = -\frac{\pi}{2} - k\pi$ 为无定义点 ($k=0, 1, 2, \dots$), 先化简 $\lim_{x \rightarrow -\frac{\pi}{2}} \frac{x(\pi+2x)}{2\cos x}$

分母为 0 时, 分子可 $\rightarrow$ ① $x_0 = -\frac{\pi}{2}$, $\lim_{x \rightarrow -\frac{\pi}{2}} \frac{x(\pi+2x)}{2\cos x} = \lim_{x \rightarrow -\frac{\pi}{2}} \frac{(\pi+2x)+2x}{2\cos x} = -\frac{\pi}{2}$, 可去间断点

能为 0. ② $x_0 \neq -\frac{\pi}{2}$, $\lim_{x \rightarrow x_0} \frac{x(\pi+2x)}{2\cos x} = \infty$, 无穷间断点.

第二讲 一元函数微分学

考点 (1) 定义 Δ (2) 计算 Δ' (3) 应用中值定理 几何应用 $10+9 \uparrow$

一、定义 $f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ (瞬时变化率)

[注] (1) 左右有别

$$\lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'_+(x_0) \text{ 右导}; \lim_{\Delta x \rightarrow 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'_-(x_0) \text{ 左导}$$

Δ 因此 $f'(x_0) \exists \iff f'_-(x_0) = f'_+(x_0)$

(2) 泛化 $\Delta x \rightarrow$ "狗"

$$f'(x_0) \triangleq \lim_{\text{狗} \rightarrow 0} \frac{f(x_0 + \text{狗}) - f(x_0)}{\text{狗}} \text{ (泛化!): 三狗完全一样}$$

(3) 换元法 (等价写法) 令 $x_0 + \Delta x = x$

$$\Rightarrow f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ (函数差值式)}$$

[例] (2018) 下列函数中在 $x=0$ 处不可导的是

(具体型考法)

A. $f(x) = |x| \sin |x|$ B. $f(x) = |x| \sin \sqrt{|x|}$ C. $f(x) = \cos |x|$ D. $f(x) = \cos \sqrt{|x|}$

[分析] Δ 只要见到 $f'(x_0)$ 即一点导数, 先写出定义

A. $f'(0) \triangleq \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{|x| \sin |x|}{x} = \lim_{x \rightarrow 0} \frac{|x|}{x} \sin |x| = 0$ $\leftarrow \frac{|x|}{x}$ 有界函数 (分段)

B. $f'(0) \triangleq \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{|x| \sin \sqrt{|x|}}{x} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{|x|}} \sin \sqrt{|x|} = 0$ $\leftarrow \Delta$ 有界函数无穷小

C. $f'(0) \triangleq \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{\cos |x| - 1}{x} = \lim_{x \rightarrow 0} -\frac{|x|^2}{2x} = 0$

D. $f'(0) \triangleq \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{\cos \sqrt{|x|} - 1}{x} = \lim_{x \rightarrow 0} -\frac{(\sqrt{|x|})^2/2}{x} = \lim_{x \rightarrow 0} -\frac{|x|}{2x}$

[例] ~~考~~ 设 $f(0)=0$, 以下极限存在能确定 $f(x)$ 在 $x=0$ 可导是

A. $\lim_{h \rightarrow 0} f(1 - \cosh h)/h^2$

B. $\lim_{h \rightarrow 0} f(1 - e^h)/h$ (抽象型考法)

C. $\lim_{h \rightarrow 0} f(h - \sinh h)/h^2$

D. $\lim_{h \rightarrow 0} f(2h) - f(h)/h \neq f'(0)$ 增量式

[分析] $f'(0) \triangleq \lim_{\text{狗} \rightarrow 0} \frac{f(\text{狗}+0) - f(0)}{\text{狗}}$ (泛化)

注: $h \rightarrow 0 \cosh h \rightarrow 1$

A. $= \lim_{h \rightarrow 0} \frac{f(1 - \cosh h + 0) - f(0)}{1 - \cosh h} \times \frac{1 - \cosh h}{h^2}$

天生的单侧极限

$= \lim_{1 - \cosh h \rightarrow 0^+} \frac{f(1 - \cosh h) - f(0)}{1 - \cosh h} \lim_{h \rightarrow 0} \frac{1 - \cosh h}{h^2} = \frac{1}{2} f'_+(0) \exists$

B. $= \lim_{h \rightarrow 0} \frac{f(1 - e^h) - f(0)}{1 - e^h} \times \frac{1 - e^h}{h}$

$= \lim_{1 - e^h \rightarrow 0} \frac{f(1 - e^h) - f(0)}{1 - e^h} \times \lim_{h \rightarrow 0} \frac{1 - e^h}{h} = -f'(0) \exists$

★ (4) 静-动原则, 不可违反此原则.

典型错误

$f(x_0 + \Delta x) - f(x_0 - \Delta x)/2\Delta x$ 可接受 $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} \neq 2f'(x_0)$

-静动

$$D. \lim_{h \rightarrow 0} \frac{f(2h) - f(h)}{h} = \lim_{h \rightarrow 0} \frac{[f(2h) - f(0)] - [f(h) - f(0)]}{h} \quad (\text{不一定可以拆开})$$

$$") \times \lim_{h \rightarrow 0} \frac{f(2h) - f(0)}{2h} - \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = 2f'(0) - f'(0) \neq f'(0)$$

取 $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$. 故 $\lim_{h \rightarrow 0} \frac{f(2h) - f(h)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$ 存在. 而不可导. 但是(1)又不可拆: $\lim_{h \rightarrow 0} \frac{f(2h)}{2h} = \lim_{h \rightarrow 0} \frac{1}{2h} = \infty$ 不 $\exists$

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} = \infty = f'(0) \text{ 不 $\exists$ }$$

故(1)又处是退化从 $\frac{0}{0} \rightarrow \infty - \infty$. 同理(2)又处 $2\infty - \infty \neq \infty$, 此时 $f'(x)$ 是一个记号

$$C. = \lim_{x \rightarrow 0} \frac{f(x - \sin x) - f(0)}{x - \sin x} \times \frac{x - \sin x}{x^2}$$

$$= \lim_{x - \sin x \rightarrow 0} \frac{f(x - \sin x + 0) - f(0)}{x - \sin x} \times \lim_{x \rightarrow 0} \frac{x^3}{6x^2} = f'(0) \cdot 0(x) \text{ 不 $\exists$ }$$

$$\Rightarrow f'(0) = \infty, \text{ 一定无界. or } f'(0) \in A; \text{ 前者不为0, 后者为0. 不一定为A, 仅为解}$$

若因为 x^4 , 此时 $f'(0) \cdot \infty = \exists \Rightarrow f'(0)$ 一定存在且一定为0. 例如 $f(x) = |x|$ 满足, 然而 $f'(0)$ 不 $\exists$.

[注]: $\frac{0}{0} > 0$ 有界但有界但 $\lim_{x \rightarrow 0} \frac{1}{x}$ 不 $\exists$ (局部有解反的聚迹)

★[例3] 若 $f(x)$ 是可导偶函数, 证明 $f'(x)$ 是奇函数. (1)

若 $f(x)$ 是可导奇函数, 证明 $f'(x)$ 是偶函数. (2)

$$[分析] \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \triangleq f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x - \Delta x) - f(x)}{-\Delta x} = -f'(x), (1) \text{ 成立}$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(-x + \Delta x) - f(-x)}{\Delta x} \triangleq f'(-x) = \lim_{\Delta x \rightarrow 0} \frac{f(x - \Delta x) + f(x)}{-\Delta x} = f'(x), (2) \text{ 成立}$$

二. 计算

1. 基本求导公式

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad (\arctan x)' = \frac{1}{1+x^2} \quad (\tan x)' = \sec^2 x$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \quad (\operatorname{arccot} x)' = -\frac{1}{1+x^2} \quad (\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x \quad \ln(x + \sqrt{x^2 + 1})' = \frac{1}{\sqrt{x^2 + 1}}$$

$$(\csc x)' = -\csc x \cot x \quad \ln(x + \sqrt{x^2 - 1})' = \frac{1}{\sqrt{x^2 - 1}}$$

2. 基本求导方法

(1) 复合函数求导

[例1] 求 $y = (\tan 2x)^{\cot \frac{x}{2}}$ 的导数

[分析] $y = \exp(\cot \frac{x}{2} \ln \tan 2x)$

$$y' = (\tan 2x)^{\cot \frac{x}{2}} \left(-\frac{1}{2} \csc^2 \frac{x}{2} \ln \tan 2x + \frac{2 \sec^2 2x}{\tan x} \cot \frac{x}{2} \right)$$

$$(uvw)' = u'vw + uv'w + uvw'$$

△ (2) 隐函数求导

显函数 $y=f(x)$ ，隐函数 $F(x,y)=0$ (一般解不出 y)

[例] 设 $y=y(x)$ 是由 $\sin xy = \ln \frac{x+e}{y} + 1$ 确定的隐函数，求 $y'(0)$ 、 $y''(0)$ 。

[分析] 设 $y=y(x)$ ，方程两端同时求导。 $y'(0)=e^2$

→ 常见计算量

对方程两边同时求导 $\cos xy (y + xy') = \frac{1}{x+e} - \frac{y'}{y} - 1$ ①

对①继续求导 $-\sin xy (y + xy')^2 + \cos xy (2y' + xy'') = -\frac{1}{(x+e)^2} - \frac{y'y - y'^2}{y^2} - 2$ ②

代 $y(0)=e^2$ 于① 得 $y'(0)=e-e^4$

代 $y'(0)=e-e^4$ 于② 得 $y''(0)=3e^6-4e^3$

(3) 对数求导法

对于多项相乘、相除、开方、乘方的式子，先取对数再求导，称为对数求导。

[例] $y = (\frac{a}{b})^x (\frac{b}{x})^a (\frac{x}{a})^b$ ， $(a,b>0)$ ，求 y' 。

[分析] 取对数 $\ln y = x \ln \frac{a}{b} + a(\ln b - \ln x) + b(\ln x - \ln a)$ ①

对①求导 $y'/y = \ln \frac{a}{b} - \frac{a}{x} + \frac{b}{x}$

$$\Rightarrow y' = (\ln \frac{a}{b} + \frac{b-a}{x}) (\frac{a}{b})^x (\frac{b}{x})^a (\frac{x}{a})^b$$

△ (4) 反函数求导

△ [例1] 设函数 $f(y)$ 的反函数 $f^{-1}(x)$ 及 $f'[f^{-1}(x)]$ 与 $f''[f^{-1}(x)]$ 均存在，且 $f'[f^{-1}(x)] \neq 0$ ，证明 $\frac{d^2 f^{-1}(x)}{dx^2} = -f''[f^{-1}(x)] / \{f'[f^{-1}(x)]\}^3$ 。 注意写法即可

[分析] $x=f(y) \Rightarrow y=f^{-1}(x)$ 注: $f^{-1}(x) = f^{-1}(f(y)) = y$

(隐函数) $\rightarrow$ 对 x 求导 $1 = f'(y) \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)}$ → 自变量为 y ，故应对 x 求导!

$$\begin{aligned} \frac{d^2 f^{-1}(x)}{dx^2} &= \frac{d^2 y}{dx^2} = d\left(\frac{dy}{dx}\right)/dx \triangleq [d(f^{-1})/dy] / [dx/dy] \\ &= -\frac{f''(y)}{f'(y)^2} \times \frac{1}{f'(y)} = -f''(y) / f'(y)^3 \end{aligned}$$

[例2] 求 $y = \frac{1}{2} \arctan \sqrt{1+x^4} + \frac{1}{4} \ln \frac{\sqrt{1+x^4}+1}{\sqrt{1+x^4}-1}$ 反函数的导数。

[分析] $y'_x = dy/dx = 1/(dx/dy) = 1/x'_y$

$$\text{令 } u = (1+x^4)^{\frac{1}{4}} \Rightarrow y = \frac{1}{2} \arctan u + \frac{1}{4} \ln \frac{u+1}{u-1}$$

$$\begin{aligned} \Rightarrow y' &= \left[\frac{1}{2} \times \frac{1}{1+u^2} + \frac{1}{4} \left(\frac{1}{u+1} - \frac{1}{u-1} \right) \right] \cdot u'_x = \frac{1}{1+u^2} \times \frac{4x^3}{4(1+x^4)^{3/4}} \\ &= -\frac{1}{x^4} \cdot \frac{4x^3}{4(1+x^4)^{3/4}} = -\frac{1}{x(1+x^4)^{3/4}} \end{aligned}$$

$$\therefore xy = \frac{1}{\sqrt{x}} = -x(1+x^4)^{3/4}$$

(5) 参数方程求导

微分方程

[例1] 设函数 $y=f(x)$ 由参数方程 $\begin{cases} x=2t+t^2 & (t>-1) \end{cases}$ 所确定, 其中 $\varphi(t)$ 具有二阶导数, 且 $\frac{d^2y}{dx^2} = \frac{3}{4(1+t)}$, 证明 $y=\varphi(t)$ $\varphi''(t) - \frac{\varphi'(t)}{1+t} = 3(1+t)$

[分析] $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\varphi'(t)}{2t+2}; \quad \frac{d^2y}{dx^2} = \frac{1}{(t+1)^2} \cdot \frac{(t+1)\varphi''(t) - \varphi'(t)}{2(t+1)} = \frac{(t+1)\varphi''(t) - \varphi'(t)}{4(t+1)^3}$$

$$\text{令 } \frac{d^2y}{dx^2} = \frac{3}{4(1+t)}, \text{ 得 } \varphi''(t) - \frac{\varphi'(t)}{1+t} = 3(1+t).$$

[例1] (2017-10-4) 设 $\begin{cases} x=t+e^t \\ y=\sin t \end{cases}$, 则 $\frac{d^2y}{dx^2}|_{t=0} =$

$$[\text{分析}] \frac{dy}{dx} = \frac{\cos t}{1+e^t}; \quad \frac{d^2y}{dx^2} = \frac{-\sin t(1+e^t) - e^t \cos t}{(1+e^t)^2} \times \frac{1}{\cos t(1+e^t)}$$

$$\text{故 } d^2y/dx^2|_{t=0} = -\frac{1}{8}.$$

△(6) 高阶导数: ① 莱布尼兹公式 ② 找规律 ③ 展开式法 → “泰勒公式”

[例1] $y=x^2 \sin 2x$, 求 $y^{(50)}$

[分析] ① 莱布尼兹公式 $= (u \cdot v)^{(n)} = \sum_{k=0}^n C_n^k u^{(k)} v^{(n-k)}$

$$\text{② 故 } y^{(50)} = x^2 \cdot (\sin 2x)^{(50)} + 50 \times 2x \cdot (\sin 2x)^{(49)} + \frac{50 \times 49}{2} \times (\sin 2x)^{(48)}$$

② 找规律, 数归法

$$(a^x)^{(n)} = (\ln a)^n \cdot a^x; \quad (\sin kx)^{(n)} = k^n \sin(kx + \frac{n\pi}{2}); \quad (\ln x)^{(n)} = (-1)^{n-1} \cdot \frac{(n-1)!}{x^n}, \quad x>0$$

$$\ln(1+x)^{(n)} = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}; \quad (\frac{1}{x+a})^{(n)} = (-1)^n \frac{n!}{(x+a)^{n+1}}$$

$$\therefore y^{(50)} = 2 \cdot x^2 \cdot \sin(2x + 25\pi) + 2^{49} \cdot 100 \sin x \sin 0x + \frac{49}{2} \pi + 2^{48} \cdot 50 \cdot 49 \cdot \sin(2x) \\ = -2^{50} \cdot x^2 \cdot \sin 2x + 2^{49} \cdot 100x \cdot \cos 2x + 2^{48} \cdot 50 \cdot 49 \cdot \sin(2x)$$

三. 中值定理 △△△, 逻辑证明与递推步骤.

1. 定理总结

(1) 涉及 $f(x)$ 的定理, 设 $f(x)$ 在 $[a, b]$ 连续. “闭区间连续”

① (有界性定理) $\exists K>0$, 使 $|f(x)| \leq K, \forall x \in [a, b]$

△② (最值定理) $m \leq f(x) \leq M$, m, M 分别为其最小, 最大值.

△③ (介值定理) 当 $m \leq \mu \leq M$, 则 $\exists \xi \in [a, b]$, 使 $f(\xi) = \mu$;

→开区间!
 Δ ④ (零点定理) 若在 $f(a) \cdot f(b) < 0$, 则 $\exists \xi \in (a, b)$ 设 $f(\xi) = 0$ (方程的根性)
 [例1] 设 $f(x)$ 在 $[a, b]$ 上连续, 证明 $\exists \xi \in [a, b]$, 使 $\int_a^b f(x) dx = (b-a)f(\xi)$
 [分析] 定积分为一个数, 即为 S .
 “积分中值定理”

结论 $\Leftrightarrow f(\xi) = \int_a^b f(x) dx / (b-a)$, 令后项为 μ . $\Rightarrow$ 介值定理

介值三部曲 ① 闭区域连续, $m \leq f(x) \leq M$ (最值定理)

Δ ② 故 $m \leq \mu \leq M$ ③ $f(\xi) = \mu$, $\xi \in [a, b]$

证: 1) $\because f(x)$ 在 $[a, b]$ 连续, $m \leq f(x) \leq M$, m, M 分别为最小、最大值

2) $m dx \leq f(x) dx \leq M dx (dx > 0)$

故 $m(a-b) \leq \int_a^b f(x) dx \leq \int_a^b M dx = M(a-b) \rightarrow$ 正向积分, $f(x) \uparrow \int_a^b f(x) \uparrow$
 “积分保号性”

即 $m \leq \mu \leq M$

3) 故 $\exists \xi \in [a, b]$, $f(\xi) = \mu$.

(2) 涉及 $f'(x)$ 的定理

⑤ 费马定理: 设 $f(x)$ 在 $x = x_0$ 处 (1) 可导 (2) 取极值 $\Rightarrow f'(x_0) = 0$.

证: 设 $f(x_0)$ 取 $x = x_0$ 取极大值

故在 $|x - x_0| < \varepsilon$, $\exists \varepsilon$, 使 $f(x_0) \geq f(x)$.

故 $f'_-(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x_0) - f(x)}{x_0 - x} \leq 0$; $f'_+(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x_0) - f(x)}{x_0 - x} \geq 0$

$\therefore$ 可导 $\therefore f'_-(x_0) = f'_+(x_0) = 0$

$\Delta \Delta$ ⑥ 罗尔定理: 设 $f(x)$ 满足三条 1) $[a, b]$ 连续 2) (a, b) 内可导 3) $f(a) = f(b)$, 则
 $\exists \xi \in (a, b)$, 使 $f'(\xi) = 0$.
 “套”

证: 必 $\exists \xi \in (a, b)$, 使 $f(\xi) \geq f(x)$ 或 $f(\xi) \leq f(x)$, 此处 $f(\xi)$ 为极值

[应用1] $f(a) = f(b) \Rightarrow f'(\xi) = 0$ “复杂化!”

Δ 方法一: 求导方法逆用法 $F(x) = f(x)e^{\varphi(x)}$

$f(\xi) = -\xi f'(\xi)$

[例1] 设 $f(x)$ 在 $[0, 1]$ 连续, $(0, 1)$ 可导, $f(1) = 0$ 证 $\exists \xi \in (0, 1)$ 令 $f(\xi) + \xi f'(\xi) = 0$

[例2] 设 $f(x)$ 在 $[0, 1]$ 上二阶可导, 且 $f(1) > 0$, $\lim_{x \rightarrow 0^+} \frac{f(x)}{x} < 0$, 证

(I) 方程 $f(x) = 0$ 在 $(0, 1)$ 上至少一实根

(II) 方程 $f(x)f''(x) + [f'(x)]^2 = 0$ 在 $(0, 1)$ 内至少两个不同的实根

[分析] $F(a) = F(b) \Rightarrow F'(\xi) = 0$ F -复杂

由 $(uv)' = u'v + v'u$ 倒着看.

1) 按例1. $f(x) + xf'(x) \xrightarrow{?} u=f(x), v=x \rightarrow F(x) = x \cdot f(x) \rightarrow F(0) = F(1)$
令 $F(x) = x \cdot f(x)$; $F(0) = F(1) \Rightarrow \exists \xi \in (0, 1)$, 令 $F'(\xi) = \xi f'(\xi) + f(\xi) = 0$

2) 令 $u=f(x), v=e^x$, 记 $F(x) = e^x \cdot f(x) \Rightarrow F'(x) = e^x [f(x) + f'(x)] \xrightarrow{?} = 0$

设 $f'(\xi) + f(\xi) = 0 \Rightarrow F'(\xi) = [e^\xi f'(\xi) + e^\xi f(\xi)] = 0 \Rightarrow F(x) = e^x \cdot f(x) \xrightarrow{?} = 0$

3) 取 $u=f(x), v=e^{\varphi(x)}$, 记 $F(x) = f(x) \cdot e^{\varphi(x)} \Rightarrow F'(x) = e^{\varphi(x)} [f'(x) + \varphi'(x) \cdot f(x)] = 0$

[例] $f(x)$ 在 $[0, 1]$ 上连续, $(0, 1)$ 可导且 $f(1) = k \int_0^{\frac{1}{2}} x e^{-x} f(x) dx, k > 1$

请证明 $\exists \xi \in (0, 1)$ 使 $f'(\xi) = (1 - \frac{1}{\xi}) f(\xi)$.

[分析] 结论 $\Leftrightarrow f'(x) + (1/x - 1)f(x) = 0 \Leftrightarrow \varphi(x) = \ln x - x$

令 $F(x) = f(x) \cdot e^{\ln x - x} = f(x) \cdot x \cdot e^{-x}$ $\rightarrow$ "见到 $\int_a^b f(x) dx \Rightarrow$ 先用积分中值定理处理再说"

$F(0) = 0$ 且 $F(1) \neq 0 \Rightarrow F(0) \neq F(1)$. 故缩小区间.

$\therefore \int_{\frac{1}{2}}^1 x \cdot e^{1-x} f(x) dx = \frac{1}{2} \cdot \eta \cdot e^{1-\eta} f(\eta), \eta \in (0, \frac{1}{2})$

$\therefore f(1) = \eta \cdot e^{1-\eta} \cdot f(\eta) \Rightarrow F(1) = \eta \cdot e^\eta f(\eta) = F(\eta)$

$\Rightarrow \exists \xi \in (\eta, 1), F'(\xi) = [f(\xi) \cdot \xi \cdot e^{-\xi}]' = 0$, 命题成立.

[例2] (I) $\because \lim_{x \rightarrow 0^+} f(x)/x < 0 \therefore \exists \delta > 0, x \in (0, \delta), f(x)/x < 0$ 根据保号性

$\Rightarrow \exists \delta > 0, x \in (0, \delta), f(x) < 0$, 不妨取 $a \in (0, \delta), f(a) < 0$

$x: f(1) > 0, f(x)$ 连续 $\therefore$ 由零点定理, $\exists c \in (a, 1) \subset (0, 1), f(c) = 0$.

解 (II) 令 $F(x) = f(x) f'(x), \Delta f(0) = \lim_{x \rightarrow 0^+} f(x) \cdot x = \lim_{x \rightarrow 0^+} \frac{f(x)}{x} \cdot x = 0 = f(c)$

$\rightarrow$ 费马定理 $\therefore \exists c' \in (0, c), f'(c') = 0$

$\therefore F(0) = F(c') = F(c), \therefore \exists \delta, \xi_1 \in (0, c'), \xi_2 \in (c', c)$ 令 $F'(\xi_1) = F'(\xi_2) = 0$

故即 $f(x) f''(x) \neq [f'(x)]^2 = 0$ 有两个实根且两者不同.

方法二. 积分化原法

① 将欲证结论多改为 x ② 积分 ③ 移项, 使等式一端为 0, 则另一端为 $F(x)$

④ 拉格朗日中值定理 $\Delta \Delta \Delta$

设 $f(x)$ 满足 1) $[a, b]$ 连续 则 $\exists \xi \in (a, b)$, 使 $f'(\xi) = \frac{f(b) - f(a)}{b - a}$

2) (a, b) 可导

(09") 证明拉格朗日中值定理.

[分析] ① $f'(x) = \frac{f(b)-f(a)}{b-a}$ ② $f(x) = \frac{f(b)-f(a)}{b-a} \cdot x \rightarrow$ "积分化原法"

③ 令 $F(x) = f(x) - \frac{f(b)-f(a)}{b-a} \cdot x$.

令 $F(x) = (b-a)f(x) - [f(b)-f(a)]x$

$F(a) = (b-a)f(a) - b f(a) + a f(b)$

$F(b) = -a f(b) + b f(a) = F(a) \Rightarrow F'(\xi) = 0 \Rightarrow f'(\xi) = \frac{f(b)-f(a)}{b-a}$ "罗尔定理"

⑧ 柯西中值定理

若 $f(x), g(x)$ 满足 1) $[a, b]$ 连续 2) $g'(x) \neq 0$, 则 $\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

2) (a, b) 可导

[证明] 不能用拉格朗日相除

① $f'(x)[g(b)-g(a)] - [f(b)-f(a)]g'(x) = 0$

② 令 $F(x) = [g(b)-g(a)]f(x) - [f(b)-f(a)]g(x)$

令 $F(x) = [g(b)-g(a)]f(x) - [f(b)-f(a)]g(x)$

$F(a) = g(b)f(a) - g(a)f(b)$

$F(b) = f(a)g(b) - g(a)f(b) = F(a) \Rightarrow F'(\xi) = 0 \Rightarrow \frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

[注] 柯西 $\xrightarrow{g'(x) \neq 0}$ 拉格朗日 $\xrightarrow{F(b)=F(a)}$ 罗尔

[例] 设 $f(x), g(x)$ 在 $[a, b]$ 二阶可导, $g''(x) \neq 0, g(b)=g(a)=f(a)=f(b)=0$

证 1) $g'(x) \neq 0, \forall x \in (a, b) \Rightarrow \exists \xi \in (a, b)$ 使 $f'(\xi)/g'(\xi) = f''(\xi)/g''(\xi)$.

[分析] 1) "反证法", 若 $\exists g'(x) = 0, x \in (a, b)$, 令 $g'(c) = 0$

$\because g(c) = g(a) \therefore \exists x_1 \in (a, c), g'(x_1) = 0 \Rightarrow g''(\xi) \neq 0, \xi \in (a, b)$

$\because g(c) = g(b) \therefore \exists x_2 \in (c, b), g'(x_2) = 0 \Rightarrow g''(\xi) = 0$

即 $g''(x_0) = 0$ 与 $g''(x)$ 在 (a, b) 不为 0 矛盾. "罗尔定理"

2) "积分化原法" 乘积作积分

① $\xi \rightarrow x, f(x)g''(x) - f''(x)g(x) = 0$ ② $f(x)g'(x) - f'(x)g(x)$

$\int f(x)g''(x)dx = \int f'(x)g(x)dx \Leftrightarrow g'(x)f(x) - \int g'(x)f(x)dx =$

$g(x)f'(x) - \int g(x)f'(x)dx \Leftrightarrow f(x)g'(x) - f'(x)g(x) = 0$

令 $F(x) = f(x)g'(x) - f'(x)g(x)$, $F(a) = F(b) = 0$ $\rightarrow$ "罗尔定理"

故 $\exists \xi \in (a, b)$ 使 $F(\xi) = 0$ 即 $f(\xi)/g(\xi) = f'(\xi)/g'(\xi)$

【应用】拉格朗日中值定理的应用

① 将 f 复杂化 $\leftarrow$ "求导变公式逆用法"

【例1】设 $f(x)$ 在 $[a, b]$ 上连续, (a, b) 内可导

证 $\exists \xi \in (a, b)$ 使 $b f(b) - a f(a) = [f(\xi) + \xi f'(\xi)](b-a)$

【分析】令 $F(x) = x \cdot f(x)$, 对 $F(x)$ 在 $[a, b]$ 用拉氏中值定理

$\Rightarrow b f(b) - a f(a) = [f(\xi) + \xi f'(\xi)](b-a)$, $\xi \in (a, b)$

△ ② 给出相对高阶条件 $\Rightarrow$ 证低阶不等式

$$f(b) - f(a) = f'(\xi)(b-a)$$

低阶 $\Rightarrow$ 高阶

【例】设 $f''(x) < 0$, $f(0) = 0$, 证明 $\forall x_1, x_2 > 0$,

有 $f(x_1 + x_2) < f(x_1) + f(x_2)$. $\rightarrow$ "多点画图分析"

【分析】不妨设 $x_1 < x_2$

$[0, x_1] \Rightarrow f(x_1) = f'(\xi_1)x_1$

$[x_2, x_1 + x_2] \Rightarrow f(x_1 + x_2) - f(x_2) = f'(\xi_2)x_1$

故 $\xi_1 \in [0, x_1] < \xi_2 \in [x_2, x_1 + x_2]$

$\because f''(x) < 0 \therefore f'(\xi_2) - f'(\xi_1) = f''(\eta)(\xi_2 - \xi_1) < 0$ 重

$\therefore f(x_1) > f(x_1 + x_2) - f(x_2)$ 即命题成立.

△ ③ 给出相对低阶条件 $\Rightarrow$ 证明高阶不等式

【例】设 $f(x)$ 二阶可导且 $f(2) > f(1)$, 且 $f(2) > \int_1^2 f(x) dx$, 证明 $\exists \xi \in (1, 2)$

使 $f'(\xi) < 0$. (2009)

$\rightarrow$ 先积分中值处理再说

【分析】 $f(2) - f(1) = f'(\xi_1) > 0$

$f(2) > \int_1^2 f(x) dx \stackrel{?}{=} 1 \cdot f(\eta)$, $\eta \in (1, 2)$

$\therefore f(2) - f(\eta) \stackrel{?}{=} (2 - \eta)f'(\xi_2) > 0 \Rightarrow f'(\xi_2) < 0$

同理 $f(\xi_2) - f(\xi_1) = f'(\xi)(\xi_2 - \xi_1) < 0 \Rightarrow f'(\xi) < 0$

④ 具体化 f . 由 $a < \xi < b \Rightarrow$ 不等式

【例】 $0 < a < b < 1$, 证 $\arctan b - \arctan a < \frac{b-a}{2ba}$

[分析] 令 $f(x) = \arctan x$, 在 $[a, b]$ 上用拉氏 $\Rightarrow \Delta \rightarrow 0 < a < \eta < b < 1$
 $f(b) - f(a) = \frac{b-a}{1+\eta^2}$, $\eta \in (a, b) \rightarrow$ "天生不等关系"
 $1+\eta^2 < 1+a^2 < a^2+b^2 < 2ab$, 即命题成立.

⑤ η 的具体表达式

[例] 设 $f(x) = \arcsin x$, η 为 $f(x)$ 在 $[0, t]$ 上的拉格朗日中值点, $0 < t < 1$, 求极限 $\lim_{t \rightarrow 0^+} \eta/t$ (4')

[分析] $\arcsin t - \arcsin 0 = \frac{1}{\sqrt{1-\eta^2}} \cdot t \Rightarrow \eta = \sqrt{1 - (\arcsin t)^2}$

令 $u = \arcsin t \Rightarrow \eta = \sqrt{1 - (\sin u/u)^2}$

$t \rightarrow 0^+$, $u \rightarrow 0^+$, 即 $\lim_{t \rightarrow 0^+} \eta/t = \lim_{u \rightarrow 0^+} \sqrt{1 - (\sin u/u)^2} / \sin u$

$= \lim_{u \rightarrow 0^+} \sqrt{u^2 - \sin^2 u} / u^2 = \sqrt{3}/3 \rightarrow \Delta u^2 - \sin^2 u \sim \frac{u^4}{3}$

[应用] 柯西中值定理应用. "分开两边讨论"

$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\eta)}{g'(\eta)}$ f -具体 g -抽象

[例] $f(x)$ 在 $[a, b]$ 上连续, 在 (a, b) 可导且

$f(x) \neq 0$, 证明 $\exists \eta \in (a, b)$, 使得 $f'(\eta)/f'(\eta) = e^b \cdot e^a / (b-a)e^\eta$

[分析] $\Rightarrow f'(\eta) = \frac{e^b - e^a}{b-a} \cdot \frac{f'(\eta)}{e^\eta}$, 其中 $f'(\eta)/e^\eta = f(b) - f(a) / e^b - e^a$

$\Delta \Delta \Rightarrow f'(\eta) = f(b) - f(a) / b - a$, 成立.

⑨ 泰勒公式 任何可导函数 $= \sum \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$

$\rightarrow$ 证明/推证某些结果

1) 带拉氏余项的公式 $f(x)$ 在 $(n+1)$ 阶可导

$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}$, ξ 在 x 与 x_0 之间

$f(x)$ 三阶可导 $\Rightarrow f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(\xi)/2 (x-x_0)^2$

当 $x_0 = 0$ 时, 则变成麦克劳林公式.

$f(x) = f(0) + f'(0)x + f''(0)x^2/2 + f'''(\xi)x^3/6$

2) 带佩亚诺余项 $\rightarrow$ "用于计算"

若 $f(x)$ 在 n 阶可导 $f(x) = f(x_0) + f'(x_0)x + f''(x_0)x^2/2 + \dots + f^{(n)}(x_0)x^n/n! + o(x^n)$

$f(x)$ 三阶可导 $\Rightarrow f(x) = f(x_0) + f'(x_0)x + f''(x_0)x^2/2 + f'''(x_0)x^3/6 + o(x^3)$

麦克劳林 $\Rightarrow f(x) = f(0) + f'(0)x + f''(0)x^2/2 + f'''(0)x^3/6 + o(x^3)$

[应用] 泰勒公式 - "信号 $f^{(n)}(\xi)$, $n \geq 2$ "

△“见到 f, f' (甚至 f'' , $n \geq 2$)、立即使用泰勒公式 (有时用拉氏余项)

△[例2] 设 $f(x)$ 在 $[0, 1]$ 上二阶可导且 $\int_0^1 f(x) dx = 0$, 则 (2018)

A. $f'(x) < 0$ 时, $f(\frac{1}{2}) < 0$ B. 当 $f'(x) < 0$ 时, $f(\frac{1}{2}) < 0$

C. $f'(x) > 0$ 时, $f(\frac{1}{2}) < 0$ D. 当 $f'(x) > 0$ 时, $f(\frac{1}{2}) < 0$

[分析] 二阶泰勒 $f(x) = f(\frac{1}{2}) + f'(\frac{1}{2})(x - \frac{1}{2}) + \frac{f''(\xi)}{2}(x - \frac{1}{2})^2$

令 $\int_0^1 \frac{f''(\xi)}{2}(x - \frac{1}{2})^2 dx = A \Rightarrow f(x) = f(\frac{1}{2}) + f'(\frac{1}{2})(x - \frac{1}{2}) + \frac{f''(\xi)}{2}(x - \frac{1}{2})^2$

$\Rightarrow \int_0^1 f(x) dx = \int_0^1 f(\frac{1}{2}) dx + \int_0^1 f'(\frac{1}{2})(x - \frac{1}{2}) dx + A \xrightarrow{1) \text{可积分}}$

$\Rightarrow 0 = f(\frac{1}{2}) + A$, 若 $f'(x) > 0 \Rightarrow A > 0 \Rightarrow f(\frac{1}{2}) < 0$ (2)可1, 2相加

四、几何应用 “三点两线”：极值、最值、拐、单调、凹凸、渐近线

1. 极值与单调性

(1) 极值的定义：必须是双侧定义，否则不称极值

1) 广义极值 $\exists x_0$ 某领域, $\forall x \in U(x_0, \delta)$, 有 $f(x) \leq f(x_0)$

则称为广义极大值。

[注] 常函数每个点都是广义极大值 “最值定理”

2) 真正极值 $\exists x_0$ 某领域, $\forall x \in \overset{\circ}{U}(x_0, \delta)$, 有 $f(x) < f(x_0)$

则称为真正极值。

[注] 若无特殊说明，按广义定义。

(2) 单调性与极值判别

1) 若 $f'(x) > 0$, $x \in I$ 则 $f(x)$ 在 I 上单调增。

2) 若 $f(x)$ 在 x_0 处连续、在 $\overset{\circ}{U}(x_0, \delta)$ 内可导

当 $x \in (x_0 - \delta, x_0)$, $f'(x) < 0$; $x \in (x_0, x_0 + \delta)$, $f'(x) > 0 \Rightarrow$ 极小值

当 $x \in (x_0 - \delta, x_0)$, $f'(x) > 0$; $x \in (x_0, x_0 + \delta)$, $f'(x) < 0 \Rightarrow$ 极大值

若两端导数不存在 $\Rightarrow$ 不变号 $\Rightarrow$ 不存在极值

3) 若 $f'(x)$ 在 $x = x_0$ 处存在, $f'(x_0) = 0$, $f''(x_0) > 0 \Rightarrow$ 极小

若 $f'(x)$ 在 $x = x_0$ 处存在, $f'(x_0) = 0$, $f''(x_0) < 0 \Rightarrow$ 极大

[注] $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + o(x - x_0)^2$

$\therefore f(x) - f(x_0) = \frac{f'(x_0)}{2}(x-x_0)^2 + o(x-x_0)^2$ "极限趋向" "改变不了正负" 高阶无穷小

若 $f'(x_0) > 0$, 则 $f(x) - f(x_0) > 0 \Rightarrow f(x) > f(x_0) \Rightarrow$ 极小 $\xrightarrow{x \rightarrow x_0}$

② 证 $f(x) - f(x_0) / (x-x_0)^2 \xrightarrow{x \rightarrow x_0} \frac{f''(x_0)}{2}$ 证 $f'(x) - f'(x_0) / 2(x-x_0) \xrightarrow{x \rightarrow x_0} \frac{f''(x_0)}{2} > 0 \Rightarrow f(x) > f(x_0) \Rightarrow$ "保号性"

△例1 证明 $f(x) = (1+\frac{1}{x})^x$ 在 $(0, +\infty)$ 单增

分析1 $f'(x) = (1+\frac{1}{x})^x [\ln(1+\frac{1}{x}) - \frac{1}{1+x}]$ "具体化"

令 $g(t) = \ln t$, $t \in (x+1, x)$, 使用拉氏公式

$\Delta \Rightarrow g(t+1) - g(t) = \ln \frac{t+1}{t} = \ln(1+\frac{1}{t}) > \frac{1}{t+1} \Rightarrow g(t+1) - g(t) - \frac{1}{t+1} > 0$

$\therefore f'(x) > 0$ 即 $f(x)$ 在 $(0, +\infty)$ 上

△例2 设 $f(x)$ 连续其 $f(x)$ 的图像如下: 则 $f(x)$ 有几个极大, 几个极小值.

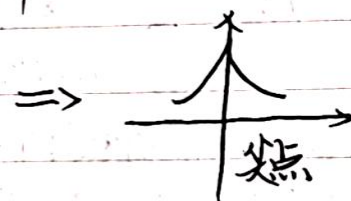
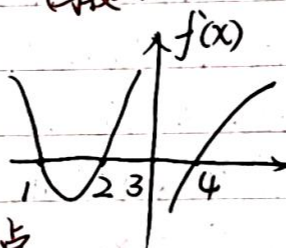
(续2)

分析 x_1 : 由 $+$ $\rightarrow$ $- \Rightarrow$ 极大

x_2, x_4 : 由 $- \rightarrow + \Rightarrow$ 极小

x_3 : 由 $+$ $\rightarrow$ $- \Rightarrow$ 极大

$\hookrightarrow$ 可以是不可导点.

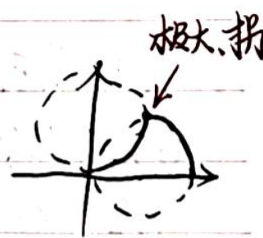


2. 凹凸性与拐点

△(1) 凹凸性: $\forall x_1, x_2 \in I$, 有

$f(x_1) + f(x_2) / 2 > f(\frac{x_1+x_2}{2}) \Rightarrow f(x)$ 是凹曲线 "U"

$f(x_1) + f(x_2) / 2 < f(\frac{x_1+x_2}{2}) \Rightarrow f(x)$ 是凸曲线 "∩"

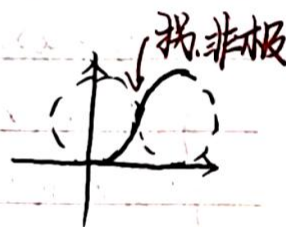


(2) 拐点: 一凹一凸连续曲线的分界点, (x_0, y_0) 且不一定可导

(3) 判别法: 设 $f(x)$ 在 I 上二阶可导

1) 若 $f''(x) > 0, \forall x \in I \Rightarrow f(x)$ 是凹 "U"

若 $f''(x) < 0, \forall x \in I \Rightarrow f(x)$ 是凸 "∩"



2) $f'(x)$ 在 x_0 左右邻域变号 $\Rightarrow (x_0, f(x_0))$ 为拐点

例1 已知曲线 $\begin{cases} x = t+1 & (t > 0) \\ y = 4t - t^2 \end{cases}$, 讨论上的凹凸性.

[分析] $dy/dx = 4-2t/2t = \frac{4}{t} - 1$

$d^2y/dx^2 = -\frac{2}{t^2} / \frac{1}{2t} = -\frac{4}{t^3} < 0 \Rightarrow$ 凸函数 "A"

[例2] 设 $f(x)=y$ 三阶可导连续, $f'(x_0)=0, f''(x_0)>0$, 证 $(x_0, f(x_0))$ 为拐点.

[分析] 构造极限 $\lim_{x \rightarrow x_0} f'(x)/(x-x_0) = \lim_{x \rightarrow x_0} \frac{f'(x)-f'(x_0)}{x-x_0}$

$\triangleq f''(x_0) > 0 \Rightarrow x < x_0, f'(x) < 0, x > x_0, f'(x) > 0$ 故拐点.

Δ [例3] 设 $y = (x-1)(x-2)^2(x-3)^3(x-4)^4$, 则其一个拐点为

A. (1,0) B. (2,0) C. (3,0) D. (4,0)

[分析] $f(x)=x^n \Rightarrow n$ 阶导后为不为0的常数.

$f'(1) \neq 0, f'(2) \neq 0, f''(3)=0$ 且 $f'''(3) \neq 0$ 由2得, 是拐点.

令 $g(x) = (x-1)(x-2)^2(x-4)^4 \Rightarrow y = (x-3)^3 g$

$y' = 3(x-3)^2 g + g'(x-3)^3; y'' = ()^1 g + ()^2 g' + ()^3 g''$

$y''' = g + ()^1 g' + ()^2 g'' + ()^3 g'''$

故 $y''(3)=0$ 且 $y'''(3) \neq 0 \Rightarrow (3, 0)$ 为拐点.

3. 渐近线 —— “极限计算能力”

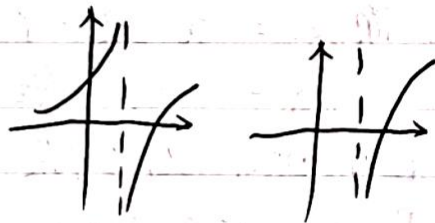
(1) 铅直渐近线

若 $\lim_{x \rightarrow x_0} f(x) = \infty$, 则称 $x=x_0$ 为 $f(x)$ 的一条铅直渐近线.

出现在无定义点 or 开区间端点, 允许单侧极限.

(2) 水平渐近线

若 $\lim_{x \rightarrow \infty} f(x) = A$, 则称 $y=A$ 为 $f(x)$ 的水平渐近线.



Δ (3) 斜渐近线

若 $\lim_{x \rightarrow +\infty} f(x)/x = a \neq 0$ 且 $\lim_{x \rightarrow +\infty} [f(x)-ax] = b$, 被称 $y=ax+b$ 是一条斜渐近线. a $\rightarrow$ 两个条件 b

[例1] $y = x + \sin \frac{1}{x}, y = x + \sin x, y = x^2 + \sin \frac{1}{x}, y = x^2 + \sin x$
问谁有渐近线.

[分析] $\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = a \neq 0 \Rightarrow f(x)$ 与 $y=ax+b$ 变化速度一样. (先决条件)
 $\lim_{x \rightarrow +\infty} x + \sin \frac{1}{x} - a = \lim_{x \rightarrow +\infty} \sin \frac{1}{x} = 0 \Rightarrow A$ 正确

【例1】 $y = e^{\frac{1}{x}} \arctan \frac{x^2+x+1}{(x-1)(x+2)}$ 有 () 条渐近线.

① 铅垂. $x=0, x=1, x=-2$ 为无定义点.

当 $x \rightarrow 0$ 时 $e^{\frac{1}{x}} \arctan \frac{x^2+x+1}{(x-1)(x+2)} \rightarrow \infty$, $x=0$ 为铅垂渐近线.

而 $\arctan x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, 故 $x \rightarrow 1, x \rightarrow -2$ 时 y 都不趋于 ∞ , 故仅一条铅垂.

② 水平. 当 $x \rightarrow \infty$ 时 $e^{\frac{1}{x}} \arctan \frac{x^2+x+1}{(x-1)(x+2)} = \frac{\pi}{4}$.

当 $x \rightarrow -\infty$ 时 $e^{\frac{1}{x}} \arctan \frac{x^2+x+1}{(x-1)(x+2)} = \frac{\pi}{4}$, 故 $y = \frac{\pi}{4}$ 为一条水平渐近线.

③ 两端均有水平, 故为无斜渐近线.

4. 最值

(1) 对 $f(x)$ 在 $[a, b]$ 上三类点

• $f'(x_0) = 0 \Rightarrow x_0$ 驻点; $f'(x)$ 不存在 $\Rightarrow x$ 不可导点; 端点 a, b .

比较 $f(x_0), f(x_1), f(a), f(b)$ 大小即可.

(2) 若在 I 上求唯一极值点, 则由实际背景 $\Rightarrow$ 此点为最值.

(3) 若在 (a, b) 端点考虑取极值.

【例1】防空洞拟建设矩形加半圆, 截面为 5m^2 , 问底

宽 x 为多少时使材料最省.

【分析】设高为 y , 则 $5 = xy + \frac{1}{2}\pi x^2 \Rightarrow y = \frac{5}{x} - \frac{\pi x}{2}$

而 $L = x + 2y + \pi x/2 = x + \frac{10}{x} - \frac{\pi x}{4} + \frac{\pi x}{2} = (1 + \frac{\pi}{4})x + \frac{10}{x}$ ($x > 0, y > 0$)

$dL/dx = (1 + \frac{\pi}{4}) - \frac{10}{x^2}$, 令 $dL/dx = 0 \Rightarrow x = \sqrt{\frac{10}{1 + \frac{\pi}{4}}} = 2\sqrt{\frac{10}{4 + \pi}}$ ($x > 0$) 唯一

由实际背景, L 又存在最值, 当 $x = \sqrt{\frac{10}{1 + \frac{\pi}{4}}}$ 时, L 最小.

且 $d^2L/dx^2 = \frac{20}{x^3} > 0$ 故 $x = \sqrt{\frac{10}{1 + \frac{\pi}{4}}}$, $d^2L/dx^2 > 0$, 故 $x = \sqrt{\frac{10}{1 + \frac{\pi}{4}}}$ 为极小值点.

【例2】求 $f(x) = nx(1-x)^n$ 在 $[0, 1]$ 上的最大值 $M(n)$ 及 $\lim_{n \rightarrow \infty} M(n)$.

【分析】 $f(0) = f(1) = 0$; $f'(x) = n(1-x)^n - n^2x(1-x)^{n-1} = n(1-x)^{n-1} [1-x + nx]$

$f'(x) = n [-(n-1)(1-x)^{n-2}(nx-x+1) + (1-x)^{n-1} \times n-1]$ “三者比较”

令 $f'(x) = 0 \Rightarrow x = \frac{1}{n+1} \in [0, 1]$; $f(\frac{1}{n+1}) = \frac{n}{n+1} (\frac{n}{n+1})^n = (\frac{n}{n+1})^{n+1} > 0$

故 $M(n) = (\frac{n}{n+1})^{n+1}$, $\lim_{n \rightarrow \infty} (\frac{n}{n+1})^{n+1} = \lim_{n \rightarrow \infty} (1 - \frac{1}{n+1})^{n+1} = e^{-1}$ (数列极限)

第三讲:一元函数积分学

核心考点 定义(强) Δ 计算 Δ 应用

一、定义

1. $\forall x \in I$, 使 $F'(x) = f(x)$ 成立, 则称 $F(x)$ 是 $f(x)$ 在 I 的一个原函数 $\rightarrow$ “宁为有计找原函数”全体原函数就叫不定积分, 记作: $\int f(x) dx = F(x) + C$ 2. 定积分 $\int_a^b f(x) dx$ 黎曼积分[小结] $\int f(x) dx$ 为函数族, $\int_a^b f(x) dx$ 为一面积值.牛顿-莱布尼兹公式 = $\int_a^b f(x) dx = F(x) \Big|_{x=a}^{x=b} = F(b) - F(a)$

二、计算(四大方法)

1. 凑微分法

(1) 基本积分公式 Δ

$$\int x^k dx = \frac{1}{k+1} x^{k+1} + C, k \neq -1; \quad \begin{cases} \int \frac{1}{x^2} dx = -\frac{1}{x} + C \\ \int \frac{1}{x} dx = \ln|x| + C \end{cases}$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C; \quad \int e^x dx = e^x + C$$

$$\Delta \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C \Leftrightarrow \int u^{-1/2} du = d(\sqrt{u})$$

$$\Delta \text{三角: } \int \sin x dx = -\cos x + C; \quad \int \cos x dx = \sin x + C \rightarrow \text{“视绝对值而不视”}$$

$$\int \tan x dx = -\ln|\cos x| + C; \quad \int \cot x dx = \ln|\sin x| + C$$

$$\int \sec^2 x dx = \tan x + C; \quad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C; \quad \int \csc x dx = \ln|\csc x - \cot x| + C$$

$$\int \sec x \tan x dx = \sec x + C; \quad \int \csc x \cot x dx = -\csc x + C$$

$$\Delta \text{ 升幂降幂: } \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C \Rightarrow \int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\Delta \int \frac{1}{\sqrt{a^2+x^2}} dx = \ln(x + \sqrt{a^2+x^2}) + C \Rightarrow \int \frac{1}{\sqrt{x^2-a^2}} dx = \ln(x + \sqrt{x^2-a^2}) + C$$

$$\int \frac{1}{1+x^2} dx = \arctan x + C; \quad \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C; \quad \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \text{ (拆项)}$$

$$\Delta \int \sqrt{a^2-x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2-x^2} + C$$

$$\text{[例1]} \quad \int \frac{dx}{(2-x)\sqrt{1-x}} \stackrel{\text{“}u\text{”}}{=} -2 \int \frac{\frac{d\sqrt{1-x}}{2-x}}{\sqrt{1-x}} = -2 \int \frac{\frac{d\sqrt{1-x}}{1+(1-x)^2}}{\sqrt{1-x}} = -2 \arctan \sqrt{1-x} + C$$

$$\text{[例2]} \quad \int \frac{x}{(2-x)\sqrt{1-x}} dx \stackrel{\text{“}u\text{”}}{=} -\int \frac{1}{2-x} d\sqrt{1-x} = -\int \frac{1}{1+(1-x)^2} d\sqrt{1-x} = -\arctan \sqrt{1-x} + C$$

[例3] $\int \frac{\cos^2 x - \sin x}{\cos x (1 + \cos x e^{\sin x})} dx$

[分析] [TH] 对于 $\int f(x) dx$, $f(x)$ 越复杂 $\Rightarrow$ 线条越多规律性越强

1) 先对复杂主要部分求导.

$\int f(x) dx = \int g(x) h(x) dx$, 若 $h'(x) = Ag(x) \Rightarrow d h(x) = Ag(x) dx$

A - 函数 $h(x)$

2) 凑微分

$\int f(x) dx = \int Ag(x) h(x) dx = \int A \cdot h(x) d h(x)$

对比题: 1) $(1 + \cos x e^{\sin x})' = e^{\sin x} (\cos^2 x - \sin x)$

$I = \int (\cos^2 x - \sin x) e^{\sin x} / e^{\sin x} \cos x (1 + \cos x e^{\sin x}) dx$

$= \int d(\cos x \cdot e^{\sin x}) / e^{\sin x} \cos x (1 + \cos x \cdot e^{\sin x})$

$= \ln(\cos x \cdot e^{\sin x} / \cos x \cdot e^{\sin x} (1 + \cos x \cdot e^{\sin x})) + C$ $\rightarrow$ 化简

[例4] 求 $\int \frac{1 - \ln x}{(x - \ln x)^2} dx$

[分析] $(x - \ln x)' = 1 - \frac{1}{x} = \frac{x-1}{x}$ $[(x - \ln x)^2]' = 2x - 2 - 2 \ln x + \frac{2 \ln x}{x}$

故创造条件不适用, 而第一步常等价变换, 常乘除 x .

原式 $= \int \frac{1 - \ln x}{x^2} \div (\frac{x - \ln x}{x})^2 dx = \dots$

$\therefore (\frac{x - \ln x}{x})' = \frac{\ln x - 1}{x^2} \therefore - \int (\frac{x - \ln x}{x})^{-2} d \frac{x - \ln x}{x} = (\frac{x - \ln x}{x})^{-1} = \frac{x}{x - \ln x} + C$

2. 换元法

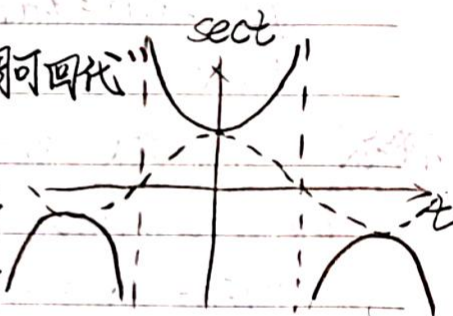
当凑微分不成功时, 考虑换元, 化简原式.

(1) 三角代换 (设 $a > 0$)

$\sqrt{a^2 - x^2} \Rightarrow$ 令 $x = a \sin t$ ($-\frac{\pi}{2} < t < \frac{\pi}{2}$) $\rightarrow$ "保单调可回代"

$\sqrt{a^2 + x^2} \Rightarrow$ 令 $x = a \tan t$ ($-\frac{\pi}{2} < t < \frac{\pi}{2}$)

$\sqrt{x^2 - a^2} \Rightarrow$ 令 $x = a \sec t$, $\begin{cases} x > 0, & 0 \leq t \leq \frac{\pi}{2} \\ x < 0, & \frac{\pi}{2} < t < \pi \end{cases}$



[注意] 若见到 $\sqrt{ax^2 + bx + c}$, 要先化为

$\sqrt{a^2 - \varphi^2(x)}$, $\sqrt{a^2 + \varphi^2(x)}$, $\sqrt{\varphi^2(x) - a^2}$, 再作三角换元

(2) 倒代换 ($x = \frac{1}{t}$) - 可用于分子次数明显低于分母次数

1) $\int \frac{1}{x^k \sqrt{a^2 - x^2}} dx$ 2) $\int \frac{1}{x^k \sqrt{a^2 + x^2}} dx$ 3) $\int \frac{1}{x^k \sqrt{x^2 - a^2}} dx$ $k = 1, 2, 3, \dots$

折后页~!

3. 分部积分法

分部分: 反三角、对数、幂函数、指数、三角 (反对幂指三)

$$d(uv)/dx = v du/dx + u dv/dx$$

$$\Rightarrow \int d(uv)/dx = \int v du/dx + \int u dv/dx = \int v du + \int u dv = uv$$

$$\Rightarrow \int u dv = uv - \int v du \Rightarrow "u, v \text{ 相对求导当} u"$$

(1) $P_n(x) \cdot e^x$, $P_n(x) \cdot \sin ax$, $P_n(x) \cdot \cos ax$. 选 $P_n(x) = u$

(2) 被积函数为 $e^{ax} \sin bx$, $e^{ax} \cos bx$. 均可 $\rightarrow$ 求导两次

(3) $P_n(x) \cdot \ln x$, $P_n(x) \cdot \arctan x$. 即选择 $\ln x$, $\arctan x$ 为 u .

[注] 分部积分公式 $\int u dv = uv - \int v du$ 推广

$$\int v u^{(n)} dx = u \cdot v^{(n)} - u' v^{(n-1)} + u'' v^{(n-2)} - \dots + (-1)^{n-1} u^{(n-1)} v + (-1)^n u^{(n)} \int v dx$$

可用表格法记忆

[例1] $\int x^3 \cos 2x dx$

$$\text{[解]} I = \frac{1}{2} \int x^3 d \sin 2x = \frac{1}{2} x^3 \sin 2x - \frac{3}{2} \int x^2 \sin 2x dx$$

$$= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} \int x^2 d \cos 2x$$

$$= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{2} \int \cos 2x \cdot x dx$$

$$= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} \int x d \sin 2x$$

$$= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x + \frac{3}{4} \int \sin 2x dx$$

$$= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x - \frac{3}{8} \cos 2x + C$$

表格法 对 $\int u v^{(n)} dx$, 可上式写成

u	$\oplus u'$	$\ominus u''$	$\oplus \dots$	$u^{(n)}$
$v^{(n)}$	$v^{(n-1)}$	$v^{(n-2)}$	$\dots$	v

① u 求导 ② $v^{(n)}$ 积分 ③ 交错相乘, 首项为正, 正负相交 ④ 最后一列相乘再积分

如上题 $x^3 \quad \oplus 3x^2 \quad \ominus 6x \quad \oplus 6 \quad \ominus 0$

$\cos 2x \quad \frac{1}{2} \sin 2x \quad -\frac{1}{4} \cos 2x \quad -\frac{1}{8} \sin 2x \quad -\frac{1}{16} \cos 2x$

$$\therefore f(x) = \frac{x^3}{2} \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x + \int \frac{3}{4} \sin 2x dx$$

$$= \frac{x^3}{2} \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x - \frac{3}{8} \cos 2x + C$$

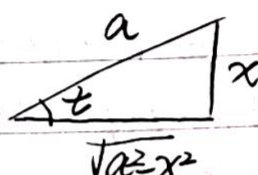
例1 求 $\int \sqrt{a^2 - x^2} dx$.

分析 原式 = $\int \sqrt{a^2 - x^2} dx$ $\xrightarrow{x=asint, -\frac{\pi}{2} < t < \frac{\pi}{2}}$ $\int a^2 \cos^2 t dt = a^2 \int \frac{1 + \cos 2t}{2} dt$

$$= \frac{a^2}{2} t + \frac{a^2}{4} \sin 2t + C$$

回代 $\frac{a^2}{2} \arcsin \frac{x}{a} + \frac{a^2}{2} \sin t \cos t + C$

$$= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2} + C$$

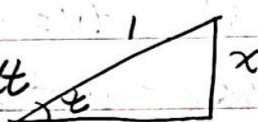


例2 求 $\int \frac{1}{x + \sqrt{1-x^2}} dx$.

分析 原式 = $\int \frac{1}{x + \sqrt{1-x^2}} dx$ $\xrightarrow{x=asint}$ $\int \frac{\cos t}{\sin t + \cos t} dt$

$$\triangleq \frac{1}{2} \int \frac{\sin t + \cos t + \cos t - \sin t}{\sin t + \cos t} dt = \frac{1}{2} t - \frac{1}{2} \ln |\sin t + \cos t| + C$$

回代 $\frac{1}{2} \arcsin x - \frac{1}{2} \ln |x + \sqrt{1-x^2}| + C$



注 对于 $\int (a_2 \sin t + b_2 \cos t) / (a_1 \sin t + b_1 \cos t) dt$

原式一般可化简为 $A(a_1 \sin t + b_1 \cos t) + B(a_1 \sin t + b_1 \cos t) \sqrt{1-x^2}$

$$\therefore \oint \text{原式} = \int A + B \ln |a_1 \sin t + b_1 \cos t| + C$$

确定 A, B $\Rightarrow \begin{cases} Aa_1 - Bb_1 = a_2 \\ Ab_1 + Ba_1 = b_2 \end{cases} \Rightarrow A, B \text{ 代入即可.}$

Δ(3) 复杂部分代换 令复杂部分 = t.

$\sqrt{ax+b} = t; \sqrt{\frac{ax+b}{cx+a}} = t; \sqrt{ae^{bx}+c} = t$ (根式代换)

注: $\arctan e^x = t$

$a^x, e^x = t; \ln x = t$ (指数、对数代换)

即可, 举重若轻

$\arcsin x, \arctan x = t$ (反三角函数代换)

例1 求 $\int \sqrt{1-x} \cdot \frac{dx}{x}$.

分析 令 $t = \sqrt{1-x} \Rightarrow x = \frac{1-t^2}{1+t^2} \Rightarrow dx = -\frac{4t}{(1+t^2)^2} dt$

注: $\int \frac{dx}{a^2 - x^2}$

代入, 原式 = $\int t \times \frac{1-t^2}{1-t^2} \times -\frac{4t}{(1+t^2)^2} dt = -\int \frac{4t^2}{1-t^4} dt$

$$= -2 \int \left(\frac{1}{t-1} + \frac{1}{t+1} \right) dt = 2 \ln \left| \frac{t-1}{t+1} \right| + 2 \arctan t + C$$

$$= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|$$

回代 $\ln \left| \frac{\sqrt{1-x} - \sqrt{1+x}}{\sqrt{1-x} + \sqrt{1+x}} \right| + 2 \arctan \sqrt{\frac{1-x}{1+x}} + C$

解法2. $I = \int \sqrt{1-x^2} / x(1+x) dx$ $\xrightarrow{x=\sin t}$ $\int \frac{\cos^2 t}{\sin t(1+\sin t)} dt$

$$= \int \frac{1}{\sin t} - 1 dt = \ln |\csc t - \cot t| - t + C$$

回代 $\ln \left| \frac{1-\sqrt{1-x^2}}{x} \right| - \arcsin x + C$

注: 正确形式可能有区别.

[例2] 求 $\int x^2 \arctan x dx$

[分析] $\arctan x$ $\begin{matrix} \textcircled{+} \frac{1}{1+x^2} \\ \textcircled{-} \frac{1}{3x^3} \end{matrix}$ 均 $P_n(x)$ 同类

$$\begin{aligned} \therefore I &= \frac{1}{3} x^3 \arctan x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{3} \int \frac{x^3+x-x}{1+x^2} dx \\ &= \frac{1}{3} x^3 \arctan x - \frac{1}{6} x^2 + \frac{1}{6} \ln(1+x^2) + C \end{aligned}$$

[例3] 求 $\int e^{2x} \sin 3x dx$

(循环积分再现)

[分析] e^{2x} $\begin{matrix} \textcircled{+} 2e^{2x} \\ \textcircled{-} \frac{1}{3} \cos 3x \end{matrix}$ $\begin{matrix} 4e^{2x} \\ \textcircled{+} -\frac{1}{9} \sin 3x \end{matrix}$

$$\therefore I = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \int e^{2x} \sin 3x dx$$

$$\Rightarrow \frac{13}{9} I = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

$$\Rightarrow I = -\frac{3}{13} e^{2x} \cos 3x + \frac{2}{13} e^{2x} \sin 3x + C$$

[例4] 设 $f(\ln x) = \frac{\ln(1+x)}{x}$, 计算 $\int f(x) dx$.

[分析] 令 $t = \ln x \Rightarrow x = e^t \Rightarrow f(t) = \ln(1+e^t) \cdot e^{-t}$

故等价于 $\int \ln(1+e^x) \cdot e^{-x} dx$

u $\ln(1+e^x)$ $\frac{e^x}{1+e^x}$
v e^{-x} $-e^{-x}$

$$\therefore I = -e^{-x} \ln(1+e^x) + \int \frac{dx}{1+e^x}$$

其中 $\int \frac{dx}{1+e^x} \xrightarrow{x=\ln t} \int \frac{dt}{t(t+1)} = \ln \frac{t}{t+1} = \ln \frac{e^x}{e^x+1} + C$

$$\therefore I = -e^{-x} \ln(1+e^x) + \ln \frac{e^x}{1+e^x} + C$$

4. 有理函数的积分

(1) 定义: 形如 $\int \frac{P_n(x)}{Q_m(x)} dx$, ($n < m$) 的积分

(2) 方法: 1) 将 $Q_m(x)$ 因式分解 2) 将 $P_n(x)/Q_m(x)$ 折成最简公式之和

(3) 拆分原则:

1) $Q_m(x)$ 分解成 $(ax+b)^k \Rightarrow k$ 项 $\Rightarrow$

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_k}{(ax+b)^k}, k=1, 2, 3, \dots$$

(2) 有理分解法 $(px^2+qx+r)^k \Rightarrow k$ 项

$$\frac{A_1x+B_1}{px^2+qx+r} + \frac{A_2x+B_2}{(px^2+qx+r)^2} + \dots + \frac{A_kx+B_k}{(px^2+qx+r)^k}, k=1,2,3,\dots$$

【例1】计算 $\int \frac{4x^2-6x+1}{(x+1)(2x-1)^2} dx$

【分析】原式 = $\frac{A}{x+1} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2} \Leftrightarrow 4x^2-6x+1 = (4A+2B)x^2 + (B+C-4A)x + (A-B+C)$

$$\therefore \begin{cases} 4A+2B=4 \\ B+C-4A=-6 \\ A-B+C=-1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=0 \\ C=-2 \end{cases}$$

$$\therefore I = \int \frac{1}{x+1} - \frac{2}{(2x-1)^2} dx = \ln|x+1| + \frac{1}{2x-1} + C$$

△另解: $\frac{4x^2-6x+1}{(x+1)(2x-1)^2} = \frac{A}{x+1} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2} \rightarrow$ “无条件成立” $\forall x$

即 $4x^2-6x+1 = A(2x-1)^2 + B(2x-1)(x+1) + C(x+1)$

取 $x=\frac{1}{2}$ 得 $-3 = \frac{3}{2}C \Rightarrow C=-2$ } “特殊值使因式为0”

取 $x=-1$ 得 $9 = 9A \Rightarrow A=1$

“比较最项系数”得 $4 = 4A+2B \Rightarrow B=0$

【例2】计算 $\int \frac{x}{x^3-x^2+x-1} dx$

【分析】 $\frac{x}{x^3-x^2+x-1} = \frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

$$\Rightarrow x = (Ax+B)(x-1) + C(x^2+1)$$

取 $x=1$ 得 $C = \frac{1}{2}$

比较 x^2 项得 $A = -\frac{1}{2}$

比较 x 项得 $B = 1/2$ (取 $x=0$)

$$\therefore I = \int \frac{1}{2(x-1)} - \frac{x/2}{x^2+1} + \frac{1/2}{x^2+1} dx$$

$$= \frac{1}{2} \ln|x-1| - \frac{1}{4} \ln|x^2+1| + \frac{1}{2} \arctan x + C$$

【例3】计算 $\int \frac{t^3-3t}{(t-1)^2(t+1)^2} dt$

【分析】知 $\frac{t^3-3t}{(t-1)^2(t+1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$

$$\Rightarrow t^3-3t = A(t-1)(t+1)^2 + B(t+1)^2 + C(t+1)(t-1)^2 + D(t-1)^2$$

令 $t=1$, $-2 = 4B \Rightarrow B = -\frac{1}{2}$

令 $t=-1$, $4D = 2 \Rightarrow D = \frac{1}{2}$

令 $t=0$, 得 $0 = -A - \frac{1}{2} + \frac{1}{2} + C \Rightarrow C = A$

比较常数项, 得 $1 = A + C \Rightarrow A = C = \frac{1}{2}$

$$\therefore I = \frac{1}{2} \int \frac{1}{t} dt - \frac{1}{2} \int \frac{1}{(t-1)^2} dt + \frac{1}{2} \int \frac{1}{t+1} dt + \frac{1}{2} \int \frac{1}{(t+1)^2} dt$$

$$= \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + \frac{1}{2(t-1)} - \frac{1}{2(t+1)} + C$$

三. 定积分的计算

$$\int_a^b f(x) dx = F(b) - F(a)$$

注: 可积的充分条件是被积函数连续

(1) 先按基本积分法求出 $F(x)$.

(2) 带入上下限时, 注意换元时的细节

$$\int_a^b f(x) dx \xrightarrow{\text{令 } x=\varphi(t)} \int_{\varphi'(a)}^{\varphi'(b)} f[\varphi(t)] \times \varphi'(t) dt \rightarrow \text{"三换"}$$

且要求 $\varphi(t)$ 连续, 并 $x = \varphi(t)$ 不超过区间 $[a, b]$

【例1】 $\int_1^4 \frac{1}{1+\sqrt{x}} dx$

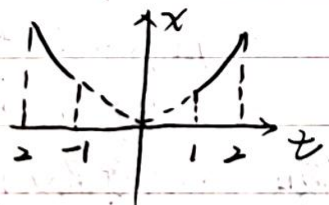
(换式代换)

【分析】 1. $\underline{\text{令 } \sqrt{x}=t}$ $\int_1^2 \frac{2t}{1+t} dt = 2 \int_1^2 \left(1 - \frac{1}{1+t} \right) dt = 2(t - \ln(t+1)) \Big|_1^2$

$$= 2 - 2\ln \frac{3}{2}$$

另解 令 $x=t^2$ $dx=2t dt$, $\sqrt{x}=|t|$

$\hookrightarrow x=1, t=\pm 1$; $x=4, t=\pm 2$.



于是四种组合 $\int_1^2, \int_{-2}^{-1}, \int_{-1}^1, \int_1^2$

Δ 而 $\int_1^2, \int_{-2}^{-1}$ 均包含 0, $\in [0, 4]$ 超过 $[1, 4]$, 故错误.

$\Delta \Delta$ 【例2】 计算 $\int_0^{\frac{\pi}{2}} \sin^n x dx$, n 为大于 1 的整数

【分析】 ① 证 $\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx \xrightarrow{\text{令 } t=\frac{\pi}{2}-x} \int_{\frac{\pi}{2}}^0 -\sin^n(\frac{\pi}{2}-t) dt = \int_0^{\frac{\pi}{2}} \cos^n t dt$$

② 递推式方法. $\{I_n\}, I_2, I_3, \dots$

Δ 分部积分法

$$n > 1 \text{ 时, } I_n = \int_0^{\frac{\pi}{2}} \sin^{n-2} x - \cos^2 x \cdot \sin^{n-2} x dx = -\int_0^{\frac{\pi}{2}} \sin^{n-1} x d\cos x$$

$$= \cos x \cdot \sin^{n-1} x \Big|_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \cos^2 x \sin^{n-2} x dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x dx = (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}, \text{ 二阶差分}$$

$$\triangle \text{ 若 } n=2k, k=0,1,2, \dots \therefore I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{1}{2} \times I_0 = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{1}{2} \times \frac{\pi}{2}$$

$$\text{若 } n=2k-1, k=1,2, \dots \therefore I_n = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{2}{3} \times I_1 = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{2}{3}$$

$$\therefore I_{10} = \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$\triangle$ 华里士公式

$$I_9 = \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1$$

$$\text{即 } \int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{n-1}{n} \times \dots \times \frac{1}{2} \times \frac{\pi}{2} & n=2k \\ \frac{n-1}{n} \times \dots \times \frac{2}{3} & n=2k-1 \end{cases} \quad (\text{Wallis equalation})$$

$$\text{[例3]} \int_{-1}^1 x^2 \sqrt{1-x^2} dx \quad \left(x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \text{不超过 } [a, b] \right)$$

$$\text{[分析]} \quad I = 2 \int_0^1 x^2 \sqrt{1-x^2} dx \xrightarrow{x=\sin t} 2 \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt$$

$$= 2 \int_0^{\frac{\pi}{2}} \sin^2 t dt - 2 \int_0^{\frac{\pi}{2}} \sin^4 t dt$$

$$= 2 \times \frac{1}{2} \times \frac{\pi}{2} - 2 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \pi/8$$

$$\text{[注]} \quad n > 1 \text{ 整数, 求 } \int_0^{2\pi} \sin^n x dx$$

[分析] 预备定理: 如 $f(x)$ 以 T 为周期且可积, 则一周期内积分与其起点无关

$$\int_0^T f(x) dx = \int_a^{a+T} f(x) dx, \forall a$$

$$\text{证: } \int_a^{a+T} f(x) dx = \int_a^T f(x) dx + \int_T^{a+T} f(x) dx$$

$$= \int_a^T f(x) dx + \int_0^a f(t+T) dt \stackrel{t=x-T}{=} \int_a^T f(x) dx + \int_0^a f(t) dt$$

$$= \int_0^T f(x) dx \quad \rightarrow \text{可拆性: } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx, \quad c \text{ 不用在 } a, b \text{ 之间}$$

如 $f(x)$ 为奇且以 T 为周期, 并可积, 求 $\int_0^T f(x) dx$.

$$\int_0^T f(x) dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) dx = 0$$

$\rightarrow$ "大题隐藏条件"

$$\text{对此题: } \int_0^{2\pi} \sin^n x dx = \int_{-\pi}^{\pi} \sin^n x dx$$

故 n 为奇, 则 $I=0$. 若 n 为正偶, $I = 2 \int_0^{\pi} \sin^n x dx$

$$\text{其中 } \int_0^{\pi} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^n x dx + \int_{\frac{\pi}{2}}^{\pi} \sin^n x dx$$

$$\triangle t=\pi-x \quad \int_0^{\frac{\pi}{2}} \sin^n x dx + \int_{\frac{\pi}{2}}^0 \sin^n(\pi-t) - dt = 2 \int_0^{\frac{\pi}{2}} \sin^n x dx$$

$$\therefore I = 4 \int_0^{\frac{\pi}{2}} \sin^n x dx = 4 \times \frac{n-1}{n} \times \dots \times \frac{1}{2} \times \frac{\pi}{2}, \quad n \text{ 为正偶}$$

, n 为奇正

$$\text{又 } \int_0^{2\pi} \cos^n x dx = \int_{-\pi}^{\pi} \cos^n x dx = 2 \int_0^{\pi} \cos^n x dx$$

$$\text{其中 } \int_0^{\pi} \cos^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx + \int_{\frac{\pi}{2}}^{\pi} \cos^n x dx$$

$$\stackrel{\text{令 } t=\pi-x}{=} \int_0^{\frac{\pi}{2}} \cos^n x dx - \int_{\frac{\pi}{2}}^0 [\cos(\pi-t)]^n dt$$

$$= \begin{cases} 2 \int_0^{\frac{\pi}{2}} \cos^n x dx, & n \text{ 为偶} \\ 0, & n \text{ 为奇} \end{cases}$$

$$\therefore \Delta \int_0^{2\pi} \cos^n x dx = \begin{cases} 4 \times \frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{1}{2} \times \frac{\pi}{2}, & n \text{ 为偶} \\ 0, & n \text{ 为奇} \end{cases} = \int_0^{2\pi} \sin^n x dx$$

$$\Delta \text{然而 } \int_0^{\pi} \cos^n x dx \neq \int_0^{\pi} \sin^n x dx$$

【例4】计算 $\int_0^6 x^2 \sqrt{6x-x^2} dx$

$$\text{【分析】 } I = \int_0^6 x^2 \sqrt{9-(x-3)^2} dx \stackrel{t=x-3}{=} \int_{-3}^3 (t+3)^2 \sqrt{9-t^2} dt$$

$$\stackrel{t=3\sin u}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 9 \times (\sin^2 u + 2\sin u + 1) \times 9 \times \cos^2 u du$$

$$= 81 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 u \cos^2 u + 2\sin u \cos^2 u + \cos^2 u) du$$

$$= 81 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 u - \sin^4 u + 2\sin u - 2\sin^3 u + \cos^2 u) du$$

$$= 162 \int_0^{\frac{\pi}{2}} (\sin^2 u - \sin^4 u + \cos^2 u) du$$

$$= 162 \times 2 \times \frac{1}{2} \times \frac{\pi}{2} - 162 \times 2 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{405}{8} \pi$$

四. 一元积分学的应用

1. 用积分表达和计算平面图形面积 —— “上下型曲线”

$y=y_1(x)$, $y=y_2(x)$, $x=a$, $x=b$ ($a < b$) 所围的面积

$$S = \int_a^b |y_1(x) - y_2(x)| dx$$

微分方法

【例】设函数 $f(x)$ 在 $[0, 1]$ 连续, $(0, 1)$ 大于 0, 并且满足 $xf'(x) = f(x) + \frac{3a}{2}x^2$, 又 $y=f(x)$ 与 $x=1$, $y=0$ 所围 S 为 2. 求 $f(x)$ 并问 a 为何值时, S 绕 x 轴旋转一周体积最小.

$$\text{【分析】 } \Rightarrow [xf'(x) - f(x)]/x^2 = [f(x)/x]' = \frac{3a}{2} \Rightarrow f(x) = \frac{3a}{2}x^2 + Cx \quad [0, 1]$$

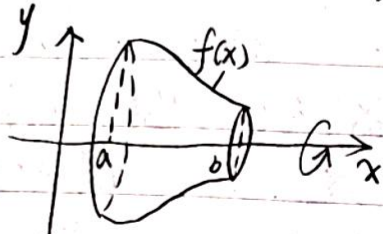
$$S = \int_0^1 (\frac{3a}{2}x^2 + Cx) dx = \frac{a}{2}x^3 + \frac{C}{2}x^2 \Big|_0^1 = \frac{a}{2} + \frac{C}{2} = 2 \Rightarrow a + C = 4$$

$$\therefore f(x) = \frac{3a}{2}x^2 + (4-a)x$$

2. 用积分计算旋转体的体积.

(1) $y=y(x)$ 与 $x=a$, $x=b$ 及 x 轴绕 x 轴周的 V

$$V_x = \int_a^b \pi f^2(x) dx$$



(2) 绕y轴旋转 — 薄壁为壳, 柱壳法

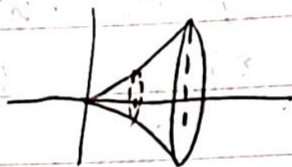
$$\Delta V_y = \int_a^b 2\pi x |f(x)| dx$$

[分析例1] 已知 $f(x) = \frac{3}{2}ax^2 + (4-a)x$

$$V_x = \int_0^1 \pi f^2(x) dx = \pi \int_0^1 \left(\frac{9}{4}a^2x^4 + 3a(4-a)x^3 + (4-a)^2x^2 \right) dx$$

$$= \pi \left[\frac{9}{20}a^2 + \frac{3}{4}a(4-a) + \frac{1}{3}(4-a)^2 \right] = \pi \left(\frac{3}{20}a^2 + \frac{1}{3}a + \frac{16}{3} \right)$$

$$dV_x/dx = \frac{a}{15} + \frac{1}{3} \quad \left\{ \begin{array}{l} \text{令 } dV_x/da = 0 \Rightarrow a = -5, \text{ 此时 } V_{x\min} \\ d^2V_x/da^2 = \frac{1}{15} > 0 \end{array} \right.$$



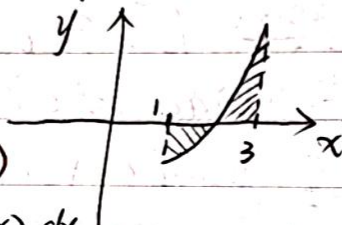
[例2] 设平面图形B由 $y = x^2 - 2x$, $y = 0$, $x = 1$, $x = 3$ 所围成, 求B绕y轴旋转一周的体积.

[分析] ① 反函数绕x轴转, 较繁琐

② $V_y = \int_1^3 2\pi x |f(x)| dx$ (柱壳法)

$$= \int_1^2 2\pi x (2x - x^2) dx + \int_2^3 2\pi x (x^2 - 2x) dx$$

$$= \frac{11}{8}\pi + \frac{43}{8}\pi = 9\pi$$



[例3] 过原点作 $y = e^x$ 的切线, 该切线与 $y = e^x$ 及x轴所围成的向x轴负向无限伸展的平面图形记D. (I) 求D的面积S. (II) D绕 $x = 1$ 旋转一周的体积.

[分析] $y' = e^x$, $y|_{x=0} = 1$, $y|_{x=0} = 1 \Rightarrow l: y = x + 1$

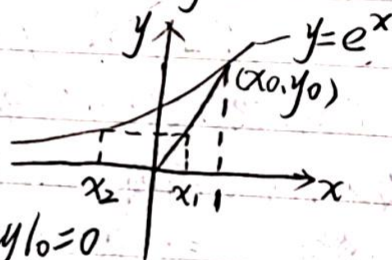
设切点 (x_0, y_0) , 则 $y_0 = e^{x_0}$, $x_0 = e^{x_0} \Rightarrow x_0 = 1 \Rightarrow l: y = ex$.

(I) $S = \int_{-\infty}^0 e^x dx + \int_0^1 (e^x - ex) dx$

$$= e - \frac{e}{2} = \frac{e}{2}$$

另解: "左右型" $S = \int_0^e \left(\frac{y}{e} - \ln y \right) dy$

$$= \frac{y^2}{2e} \Big|_0^e - (y \ln y - y) \Big|_0^e = \frac{e}{2} \quad \text{注: } y \ln y \Big|_0 = 0$$



(II) $V_{x=1} = \int_{-\infty}^1 2\pi(1-x)e^x dx - \int_0^1 2\pi(1-x)ex dx \rightarrow \lim_{y \rightarrow 0^+} y \ln y = 0$, 综合型

$$= 2\pi \left(e - \frac{e}{2} \right) = \frac{5}{2}\pi e$$

4.3. 用积分表达和计算 $f(x)$ 的平均值

$f(x)$ 在 $[a, b]$ 上的平均值 $\bar{y}(x) = \frac{\int_a^b f(x) dx}{b-a}$ — "积分中值定理"

[例1] 函数 $y = \ln x$ 在区间 $[1, e]$ 上的 $\bar{y}(x)$

[分析] $\bar{y}(x) = \int_1^e \ln x dx / (e-1) = \frac{(x \ln x - x) \big|_1^e}{e-1} = 1/(e-1)$

五. 一元函数的综合题

例1] 设 $f(x), g(x)$ 在对称区间 $[-a, a]$ 上连续, $g(x)$ 为偶且 $f(x) + f(-x) = A$, A 为常数, 证明 $\int_{-a}^a f(x)g(x)dx = A \int_0^a g(x)dx$.

(II) 求定积分 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x| \arctan e^x dx$.

[分析] (I) $\int_{-a}^a f(x)g(x)dx = \int_{-a}^0 f(x)g(x)dx + \int_0^a f(x)g(x)dx$

其中 $\int_{-a}^0 f(x)g(x)dx \xrightarrow{x=-t} -\int_a^0 f(-x)g(-x)dx = -\int_0^a [A-f(x)]g(x)dx$

$\therefore \int_{-a}^a f(x)g(x)dx = \int_0^a [A-f(x)]g(x)dx + \int_0^a f(x)g(x)dx = A \int_0^a g(x)dx$

(II) 令 $f(x) = \arctan e^x$, $g(x) = |\sin x|$.

其中 $f(x) + f(-x) = \arctan e^x + \arctan \frac{1}{e^x} = \pi/2$, $g(x)$ 为偶

$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)g(x)dx = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \sin x dx = \frac{\pi}{2}$

$\hookrightarrow$ 证明: 令 $F(x) = f(x) + f(-x)$, $F'(x) = \frac{e^x}{1+e^{2x}} - \frac{e^{-x}}{1+e^{-2x}} = 0 \leftarrow \Delta$ 导数的使用

同理 $F(0) = 2 \arctan 1 = \frac{\pi}{2}$, 此等式成立. 即 $A = \pi/2$.

例2] (I) 设 $f(x), g(x)$ 在 $[a, b]$ 上连续且 $g(x)$ 不变号, 证明 $\exists \xi \in [a, b]$, 使 $\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx$. (推广积分中值定理, 可直接使用)

(II) 求 $\lim_{n \rightarrow \infty} \int_0^1 \frac{1}{1+x^n} dx$.

[分析] (I) 若 $g(x) = 1$ 且不变号 $\Rightarrow \int_a^b f(x)dx = f(\xi)(b-a)$ 积分中值定理 $\Rightarrow$ 介值

① 若 $g(x) \equiv 0$, 显然成立.

② 不妨假设 $g(x) > 0, \forall x \in [a, b]$. 即证 $f(\xi) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$

1) $m \leq f(x) \leq M$

2) $mg(x) \leq f(x)g(x) \leq Mg(x) \xrightarrow{\Delta \text{ 不变号}} \int_a^b (b-a)mg(x)dx \leq \int_a^b f(x)g(x)dx \leq \int_a^b Mg(x)dx$

$\Leftrightarrow m \leq \mu \leq M$

3) 由介值定理得 $\exists \xi \in [a, b]$, 令 $f(\xi) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$.

③ 同理可证 $g(x) < 0$, 依旧成立.

1) $m \leq f(x) \leq M$ 2) $mg(x) \geq f(x)g(x) \geq Mg(x) \xrightarrow{\Delta \text{ 不变号}} \int_a^b mg(x)dx \geq \int_a^b f(x)g(x)dx \geq \int_a^b Mg(x)dx$

$\Leftrightarrow m \leq \mu \leq M \Rightarrow$ 由介值定理, $\exists \xi \in [a, b]$, 使 $f(\xi) = \mu$.

$$(II) \int_0^1 \frac{1}{1+x} \cdot x^n dx = \frac{1}{1+\xi} \int_0^1 x^n dx = \frac{1}{(n+1)(\xi+1)}, \xi \in [0, 1]$$

$$\therefore \frac{1}{2(n+1)} \leq \frac{1}{(n+1)(\xi+1)} \leq \frac{1}{n+1} \quad \text{"从简单下手"}$$

$\hookrightarrow$ 由 n 决定.

$$n \rightarrow \infty \quad 0 = \frac{1}{2(n+1)} \leq \frac{1}{(n+1)(\xi+1)} \leq 0 = \frac{1}{n+1}, \text{夹逼准则}$$

$$\therefore \int_0^1 \frac{x^n}{1+x} dx = 0$$

第四讲 多元函数微分学

核心考点 概念 4' 计算 极最值 10'

一、概念

1. 设 $f(x, y)$ 的定义域为 D , $P_0(x_0, y_0)$ 为 D 的聚点

$\forall \varepsilon > 0, \exists \delta > 0$, 当 $P(x, y) \in D \cap \dot{U}(P_0, \delta)$ 时, 恒有

$$|f(x, y) - A| < \varepsilon \Rightarrow \lim_{x \rightarrow x_0, y \rightarrow y_0} f(x, y) = A$$

若 $D \setminus \{x, y \neq 0\}$ $P_0(0, 0) \therefore D \cap \dot{U}(P_0, \delta) \Rightarrow$

[注] 求 $\lim_{xy \rightarrow 0} \frac{\sqrt{xy+1} - 1}{\sqrt{xy}} xy$

① 其他教材 $P \in \dot{U}(P_0, \delta)$ 即同, 此极限不存在, 唯一性

② 同济教材 $P \in D \cap \dot{U}(P_0, \delta)$, 其中 $D \setminus \{x, y \neq 0\}$ $\Delta \lim_{(x, y) \rightarrow (0, 0)} \frac{\sin(xy)^{1/3}}{x^2+y^2}$

$\therefore$ 原式 $= \lim_{xy \rightarrow 0} \frac{xy}{xy(\sqrt{xy+1}+1)} = \frac{1}{2}$ (平方差公式) $\star$ $= -\frac{1}{6} \star$

2. 连续性

若 $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$, 则称 $f(x, y)$ 在 (x_0, y_0) 处连续

[注] 若 $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) \neq f(x_0, y_0)$, 则不连续, 不讨论间断模型

3. 偏导数 (必考) $z = f(x, y)$

$$f'_x(x_0, y_0) \triangleq \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)}$$

$$\left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} = f'_y(x_0, y_0)$$

$$\triangleq \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

例 1 设 $f(x, y) = e^{\sqrt{x}y}$, 求 $f'_x(0, 0), f'_y(0, 0)$

[分析] 严格按定义执行

$\star$ 见到 $f'_x(x_0, y_0), f'_y(x_0, y_0)$ 先写定义再说

$$f'_x(0, 0) \triangleq \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{e^{-1/\Delta x} - 1}{\Delta x} \quad (\text{一元极限})$$

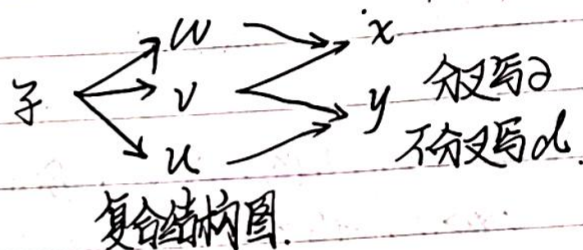
故极限不存在, 唯一性

同理 $f'_y(0, 0) \triangleq \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{e^{0} - 1}{\Delta y} = 0 \star$

例 1 设 $f(x, y) = \begin{cases} 0, & xy=0 \\ \frac{1}{xy} \sin(x^2y), & xy \neq 0 \end{cases}$ 则 $f'_x(0, 1) =$

[分析] $f'_x(0, 1) \triangleq \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 1) - f(0, 1)}{\Delta x}$

$$= \lim_{\Delta x \rightarrow 0} \sin(\Delta x)^2 / (\Delta x)^2 = 1 \quad \times$$



二. 计算 (必考) 一元函数微分法

1. 链式求导规则

设 $z = f(u, v, w)$, $u = u(x, y)$, $v = v(x, y)$, $w = w(x)$

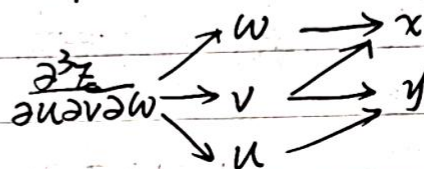
称 x, y 为自变量, u, v, w 为中间变量, z 为因变量.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial x}; \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

2. 高阶偏导数

不论 z 对谁求导, 无论 z 求几阶导, 求导后的偏导数仍然与 z 相同的复合结构

例1 设 $F(u, v)$ 对变元 u, v 具有二阶连续偏导数, 并设 $z = F(\frac{y}{x}, x^2 + y^2)$, 求 $\frac{\partial^2 z}{\partial x \partial y}$.



按考研答案写符号记号.

$$\frac{\partial z}{\partial x} = F_1'(-\frac{y}{x^2}) + F_2' \cdot 2x = -\frac{y}{x^2} F_1' + 2x F_2' \quad (2')$$

Δ 在具有二阶连续偏导数时.

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &\triangleq \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(-\frac{y}{x^2} F_1' + 2x F_2' \right) \\ &= \frac{\partial F_1'}{\partial y} \cdot \left(-\frac{y}{x^2}\right) + F_1' \frac{\partial}{\partial y} \left(-\frac{y}{x^2}\right) + 2x \frac{\partial F_2'}{\partial y} \\ &= \frac{\partial F_1'}{\partial y} \cdot \left(-\frac{y}{x^2}\right) - F_1' / x^2 + 2x \frac{\partial F_2'}{\partial y} \end{aligned}$$

$$F_1' < \begin{matrix} 1 \\ 2 \end{matrix} \begin{matrix} (\frac{y}{x}) \\ (x^2 + y^2) \end{matrix} \begin{matrix} x \\ y \end{matrix}$$

$$F_2' < \begin{matrix} 1 \\ 2 \end{matrix} \begin{matrix} (\frac{y}{x}) \\ (x^2 + y^2) \end{matrix} \begin{matrix} x \\ y \end{matrix}$$

其中 $\frac{\partial F_1'}{\partial y} = F_{11}'' \cdot \frac{1}{x} + F_{12}'' \cdot 2y$

$$\frac{\partial F_2'}{\partial y} = F_{12}'' \cdot \frac{1}{x} + F_{22}'' \cdot 2y$$

$$\text{代入得 } \frac{\partial^2 z}{\partial x \partial y} = -\frac{y}{x^3} F_{11}'' + 2 \left(\frac{x^2 + y^2}{x^2} \right) F_{12}'' + 4xy F_{22}'' - \frac{F_1'}{x^2} \quad \times$$

例1 设 $f(u, v)$ 二阶偏导数连续, $z = f(x^2y, \cos x \sin y)$, 求 $\frac{\partial^2 z}{\partial x \partial y}$.

$$\text{例析1 } \frac{\partial z}{\partial x} = 2xy F_1' - \sin x \sin y F_2'$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= 2x F_1' + 2xy \left(\frac{\partial F_1'}{\partial y} \right) - \sin x \cos y F_2' - \sin x \sin y \frac{\partial F_2'}{\partial y} \\ &= 2x F_1' - \sin x \cos y F_2' + 2xy (x^2 F_{11}'' + \cos x \cos y F_{12}'') - \sin x \sin y (x^2 F_{12}'' + \cos x \cos y F_{22}'') \end{aligned}$$

$$= 2xF'_1 - \sin x \cos y F'_2 + 2x^2y F''_{11} + (2xy \cos x \cos y - x^2 \sin x \cos y) F''_{12} - \sin x \cos x \sin y \cos y F''_{22} \quad *$$

三. 多元函数的极最值 (必考)

1. 无条件的极值 $z = f(x, y)$

(1) 必要条件

设 $z = f(x, y)$ 在 (x_0, y_0) 处, 一阶偏导

存在且取极值 $\Rightarrow f'_x(x_0, y_0) = f'_y(x_0, y_0) = 0$

费马定理: 若 $f(x)$ 在 x_0 处取极值且存在 $f'(x_0) \Rightarrow f'(x_0) = 0$

[注] 适用于三元及以上 (常考 3-5 元) $\rightarrow$ 拉氏乘数法

(2) 充分条件

$$\left. \begin{aligned} f''_{xx}(x_0, y_0) &= A \\ f''_{xy}(x_0, y_0) &= B \\ f''_{yy}(x_0, y_0) &= C \end{aligned} \right\} \Rightarrow \Delta = B^2 - AC = \begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

$$\Delta = 0 \quad \text{此法无效 凹凸线 A U}$$

$$\left. \begin{aligned} &> 0 \Rightarrow \text{不是极值点} \uparrow \\ &< 0 \end{aligned} \right\} \begin{aligned} &A > 0 \Rightarrow \text{极小值} \\ &A < 0 \Rightarrow \text{极大值} \end{aligned}$$

[注] 只适用于二元. 必考.

[例] 设 $z = z(x, y)$, 由方程 $x^2 - 6xy + 10y^2 - 2yz - z^2 + 32 = 0$ 确定, 讨论 $z(x, y)$ 的极大值与极小值. (隐式表达式) (*)

[分析] 无条件极值解题程序

① 给出 $z(x, y)$ 显式 ② 若 $z = z(x, y)$ 由 $F(x, y, z) = 0$ 确定 隐式
程序 1° 写 $z'_x \triangleq 0; z'_y \triangleq 0 \Rightarrow p_0 (i=1, 2, \dots)$ 候选点

2° 写 $z''_{xx}, z''_{xy}, z''_{yy}$, 分别代入 p_0 , 则 A_i, B_i, C_i .

令 $\Delta_i = B_i^2 - A_i C_i$, 以此来判别结果.

1° (*) 对 x 偏导 $2x - 6y - 2yz'_x - 2zz'_x = 0$, 即 $z'_x = \frac{x-3y}{y+z}$ (1)

(*) 对 y 偏导 $-6x + 20y - 2z - 2zz'_y = 0 - 2yz'_y = 0$ (2)

即 $-6x + 20y - 2z - (2z + 2y)z'_y = 0$

令 $z'_x = 0$ 且 $z'_y = 0$, 得 $\begin{cases} 2x - 6y = 0 \\ -3x + 10y - z = 0 \end{cases}$ 代入 (*)

$$9y^2 - 18y^2 + 10y^2 - 2y^2 - y^2 + 32 = 0 \Rightarrow y^2 = 16, y = \pm 4$$

$$\therefore P_1: \begin{cases} x_1 = 12 \\ y_1 = 4 \\ z_1 = 4 \end{cases} \quad \text{or } P_2: \begin{cases} x_1 = -12 \\ y_1 = -4 \\ z_2 = -4 \end{cases} \quad (\text{备选点})$$

$$2^\circ \quad \begin{cases} x - 3y - (y+z)z'x = 0 & \dots\dots (1) \\ -3x + 10y - z - (y+z)z'y = 0 & \dots\dots (2) \end{cases}$$

$$\text{对 (1) 求 } x \text{ 偏导} \quad 1 - (z'x)^2 - z''_{xx}(y+z) = 0 \quad \dots\dots (3)$$

$$\text{对 (1) 求 } y \text{ 偏导} \quad -3 - (1+z'y)z'x - (y+z)z''_{xy} = 0 \quad \dots\dots (4)$$

$$\text{对 (2) 求 } y \text{ 偏导} \quad 10 - z'y - (1+z'y)z'y - (y+z)z''_{yy} = 0 \quad \dots\dots (5)$$

$$\therefore z'x = z'y = 0 \quad \therefore z''_{xx} = 1/(y+z)$$

$$z''_{xy} = -3/(y+z)$$

$$z''_{yy} = 10/(y+z)$$

$$\text{代入 } P_1(12, 4, 4) \Rightarrow \begin{cases} z''_{xx} = 1/8 = A_1 \\ z''_{xy} = -3/8 = B_1 \\ z''_{yy} = 10/8 = C_1 \end{cases} \Rightarrow \begin{cases} \Delta_1 = B_1^2 - A_1 C_1 = \frac{9}{64} - \frac{10}{64} = -\frac{1}{64} < 0 \\ A_1 = \frac{1}{8} > 0 \end{cases}$$

$\therefore$ 一定为 $(12, 4)$ 为极小值点

$z = 4$, 为极小值

$$\text{代入 } P_2(-12, -4, -4) \Rightarrow \begin{cases} z''_{xx} = -1/8 = A_2 \\ B_2 = 3/8 \\ C_2 = -1/8 \end{cases} \Rightarrow \begin{cases} \Delta_2 = B_2^2 - A_2 C_2 = -\frac{1}{64} < 0 \\ A_2 = -\frac{1}{8} < 0 \end{cases}$$

$\therefore (-12, -4)$ 为极大值点

$z = -4$ 为极大值 $\times$

例1 求二元函数 $f(x, y) = x^2(2 + \frac{y^2}{x^2}) + y \ln y$ 的极值

分析 无条件极值

$(y > 0)$

$$1^\circ \quad z'_x = 4x + 2y^2/x = 2x(2 + y^2/x^2)$$

$$z'_y = 2x^2y + 1 + \ln y$$

$$\text{令 } \begin{cases} z'_x = 2x(2 + y^2/x^2) = 0 \\ z'_y = 2x^2y + 1 + \ln y = 0 \end{cases} \Rightarrow P(0, \frac{1}{e})$$

$$z'_y = 2x^2y + 1 + \ln y = 0$$

$$2^0 \quad z''_{xx} = 4 + 2y^2 \quad \text{代入 } p(0, \frac{1}{e}) \text{ 得 } A = 4 + \frac{2}{e^2}$$

$$z''_{xy} = 4xy \quad B = 0$$

$$z''_{yy} = 2x^2 + \frac{1}{y} \quad C = e$$

$$\text{此时 } \Delta = B - 4AC = -4e(4 + \frac{2}{e^2}) < 0$$

$\therefore A > 0 \therefore (0, \frac{1}{e})$ 为极小值点, 极小值为 $-\frac{1}{e}$ *

2. 条件极值(最值)

提法: 求目标函数 $u = f(x, y, z)$ 在约束条件 (s.t.) $\begin{cases} \varphi(x, y, z) = 0 \\ \psi(x, y, z) = 0 \end{cases}$ 下的极值.

拉氏乘数法:

5个独立变量

五元函数

(λ, μ 可取 0)

(1) 构造辅助函数 $F(x, y, z, \lambda, \mu) = f(x, y, z) + \lambda \varphi(x, y, z) + \mu \psi(x, y, z)$

(2) 令 $F'_x = 0, F'_y = 0, F'_z = 0, F'_\lambda = 0, F'_\mu = 0$ (必要条件)

(3) 解方程组 $\Rightarrow p_i \Rightarrow u_i$ 比较 $\Rightarrow$ 取最大最小为最大值最小值 (难点)

例1 求函数 $u = xy + \frac{2}{3}yz$ 在约束条件 $x^2 + y^2 + z^2 = 10$ 下的最值.

分析: 设 $F(x, y, z, \mu) = xy + \frac{2}{3}yz + \mu(x^2 + y^2 + z^2 - 10)$

$$F'_x = y + 2\mu x \quad \text{① 令 } F'_x = F'_y = F'_z = F'_\mu = 0$$

$$F'_y = x + \frac{2}{3}z + 2\mu y \quad \text{② 得}$$

$$F'_z = \frac{2}{3}y + 2\mu z \quad \text{③}$$

$$F'_\mu = x^2 + y^2 + z^2 - 10 \quad \text{④}$$

解法: 消元代入法 令 $\mu = 0, y = 0, x + 2z = 0$

得 $(2\sqrt{5}, 0, \sqrt{5}, 0)$ 与 $(-2\sqrt{5}, 0, -\sqrt{5}, 0)$

① $x +$ ② $y +$ ③ z , 得 $xy + xy + 2yz + 2yz + 4\mu(x^2 + y^2 + z^2) = 0$

$$\therefore 10\mu + xy + 2yz = 0$$

$$\therefore z = 2x \quad \therefore xy = -2\mu \quad \text{则 ② } 5x + 2\mu y = 0 \Rightarrow \mu = -\frac{5x}{2y}$$

$$\therefore -25x^2 - 25x^2 \Rightarrow x(y^2 - 5) = 0 \Rightarrow y = \pm\sqrt{5}$$

$\therefore (-1, \sqrt{5}, -2) \quad (1, \sqrt{5}, 2)$ 六个候选点.

$(-1, -\sqrt{5}, -2) \quad (1, -\sqrt{5}, 2)$

$$f_1 = -5\sqrt{5} - 4\sqrt{5} = -9\sqrt{5}; f_2 = 8 + 4\sqrt{5} = 5\sqrt{5}; f_3 = 5\sqrt{5}$$

$$f_4 = -5\sqrt{5}; f_5 = 0; f_6 = 0$$

$\therefore u = xy + 2yz$ 在 $x^2 + y^2 + z^2 = 10$ 下最大值为 $5\sqrt{5}$, 最小值为 $-5\sqrt{5}$. *

△解法二. 对于 $xy + 2yz + 10\lambda = 0$ 时, 即 $u + 10\lambda = 0$ 即 $u = -10\lambda$

若求 λ 的所有取值 $\Rightarrow u_{\max}, \min$.

$$\text{则此时} \begin{cases} y + 2\lambda x = 0 \\ x + 2\lambda y + 2\lambda z = 0 \\ 2y + 2\lambda z = 0 \end{cases} \begin{matrix} \text{关于 } x, y, z \text{ 齐} \\ \text{次线性方程} \\ \xleftrightarrow{x^2+y^2+z^2=10} \end{matrix} \begin{pmatrix} 2\lambda & 1 & 0 \\ 1 & 2\lambda & 2 \\ 0 & 2 & 2\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

又 x, y, z 不能同时取 0, $\therefore$ 此齐次方程一定有非零解.

$$\Rightarrow \begin{vmatrix} 2\lambda & 1 & 0 \\ 1 & 2\lambda & 2 \\ 0 & 2 & 2\lambda \end{vmatrix} = 0 \Rightarrow \lambda(4\lambda^2 - 5) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = \frac{\sqrt{5}}{2} \quad \lambda_3 = -\frac{\sqrt{5}}{2}$$

$$u_1 = 0 \quad u_2 = 5\sqrt{5} \quad u_3 = -5\sqrt{5}$$

$\therefore u$ 在 $x^2 + y^2 + z^2 = 10$ 下 $u_{\max} = 5\sqrt{5}$ $u_{\min} = -5\sqrt{5}$ *

四. 例题

例1 某公司可通过电台与报纸两种做广告, 销售收入 R 与电台广告费 x_1 及报纸广告费 x_2 关系如下 $R = 15 + 4x_1 + 32x_2 - 8x_1x_2 - 2x_1^2 - 10x_2^2$.

(1) 在不限广告费条件下, 最优广告策略.

(2) 若广告费为 1.5 万元, 求相应的最优策略.

分析 (1) 多元函数的极值.

$$R'_{x_1} = 14 - 8x_2 - 4x_1; \quad R'_{x_2} = 32 - 8x_1 - 20x_2$$

$$R''_{x_1x_1} = -8; \quad R''_{x_1x_2} = -8; \quad R''_{x_2x_2} = -20$$

$$\Delta = (R''_{x_1x_2})^2 - 4R''_{x_1x_1} \times R''_{x_2x_2} < 0, \quad R''_{x_1x_2} < 0, \text{ 故为极大值.}$$

$$\text{令} \begin{cases} 14 - 8x_2 - 4x_1 = 0 \\ 32 - 8x_1 - 20x_2 = 0 \end{cases} \Rightarrow$$

$$32 - 8x_1 - 20x_2 = 0$$

错误在于 $R \neq U$! 应用题.

(1) 设净利润函数 $z = f(x_1, x_2) = R - x_1 - x_2$

1° $f'_{x_1} = 13 - 8x_2 - 4x_1$; $f'_{x_2} = 31 - 8x_1 - 20x_2$

$f'_{x_i} = 0 \Rightarrow \begin{cases} 13 - 8x_2 - 4x_1 = 0 \\ 31 - 8x_1 - 20x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0.75 \\ x_2 = 1.25 \end{cases}$

[注] $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \\ x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \end{cases}$ (克拉默法则)

对比题 $\begin{cases} 4x_1 + 8x_2 = 13 \\ 8x_1 + 20x_2 = 31 \end{cases}$ 得 $\begin{vmatrix} 4 & 8 \\ 8 & 20 \end{vmatrix} = 16$; $\begin{vmatrix} 13 & 8 \\ 31 & 20 \end{vmatrix} = 12$; $\begin{vmatrix} 4 & 13 \\ 8 & 31 \end{vmatrix} = 20$

$\therefore x_1 = 0.75 = A$ $x_2 = 1.25 = B$

2° $f''_{x_1x_1} = -4$; $f''_{x_1x_2} = -8$; $f''_{x_2x_2} = -20 = C$

$\Delta = (-8)^2 - 4 \times 20 < 0$, $\therefore$ 所求极值为极大值, $A < 0$
 $= B^2 - AC$

3° $(0.75, 1.25)$ 为极大值点, 根据实际问题, 此点为最大值点

即投入电台 0.75 万元, 报纸 1.25 万元时利润最大.

(2) 条件极值问题

1° 构造 $F(x_1, x_2, \lambda) = 15 + 13x_1 + 31x_2 - 8x_1x_2 - 2x_1^2 - 10x_2^2 + \lambda(x_1 + x_2 - 1.5)$

2° 求 $F'_{x_1} = 13 - 4x_1 - 8x_2 + \lambda = 0 \Leftrightarrow \begin{vmatrix} 1 & 1 & 0 \\ 4 & 8 & -1 \\ 8 & 20 & -1 \end{vmatrix} = 0$
 $\begin{cases} F'_{x_2} = 31 - 8x_1 - 20x_2 + \lambda = 0 \\ F'_{\lambda} = x_1 + x_2 - 1.5 = 0 \end{cases}$

$= \begin{vmatrix} 4 & 8 & -1 \\ 4 & 12 & 0 \\ 1 & 1 & 0 \end{vmatrix} = -1 \times (-1)^{3+1} \times \begin{vmatrix} 4 & 12 \\ 1 & 1 \end{vmatrix} = 8$, 拉格朗日法可用

$\therefore \begin{vmatrix} 13 & 8 & -1 \\ 31 & 20 & -1 \\ 1.5 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 13 & 8 & -1 \\ 18 & 12 & 0 \\ 1.5 & 1 & 0 \end{vmatrix} = -1 \times (-1)^{3+1} \times \begin{vmatrix} 13 & 8 \\ 1.5 & 1 \end{vmatrix} = 0$

$$\begin{vmatrix} 4 & 13 & -1 \\ 8 & 31 & -1 \\ 1 & 1.5 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 13 & -1 \\ 4 & 18 & 0 \\ 1 & 1.5 & 0 \end{vmatrix} = -1 \times (-1)^4 \times \begin{vmatrix} 4 & 18 \\ 1 & 1.5 \end{vmatrix} = 12$$

$$\begin{vmatrix} 4 & 8 & 13 \\ 8 & 20 & 31 \\ 1 & 1 & 1.5 \end{vmatrix}$$

无需计算

$$\therefore x_1 = 0$$

$$x_2 = 1.5$$

∴ 在高广告费1.5万下, 全用于报纸, 利润最大

#

第五讲 二重积分

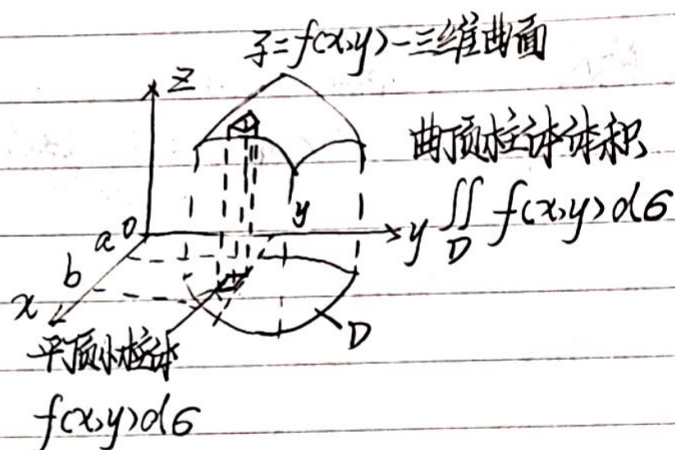
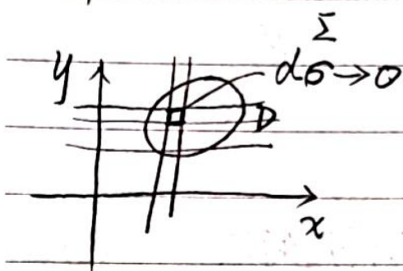
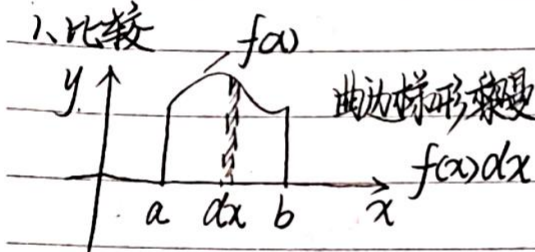
4' + 10'

核心考点 1. 概念 4'/10'

2. 计算 10'

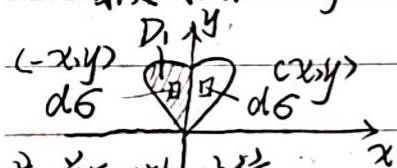
一、概念

1. 比较

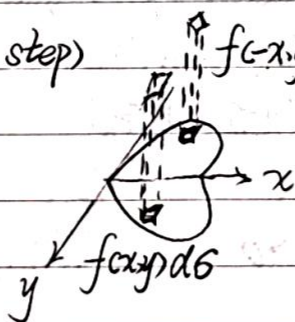


Δ 2. 对称性 (必考)

(1) 普遍对称性 (first step)



若D关于y轴对称



偶倍:

$$1^{\circ} \text{ 若 } f(x,y) = f(-x,y), \text{ 如 } \cos x \sin y \\ \Rightarrow \iint_D f(x,y) d\sigma = 2 \iint_{D_1} f(x,y) d\sigma$$

$$2^{\circ} \text{ 若 } f(x,y) = -f(-x,y), \text{ 如 } \sin x \cos y \\ \Rightarrow \iint_D f(x,y) d\sigma = 0 \text{ 奇零}$$

同理若D关于x轴对称

$$1^{\circ} \text{ 若 } f(x,y) = f(x,-y), \iint_D f(x,y) d\sigma = 2 \iint_{D_1} f(x,y) d\sigma$$

$$2^{\circ} \text{ 若 } f(x,y) = -f(x,-y), \iint_D f(x,y) d\sigma = 0$$

若D关于原点对称

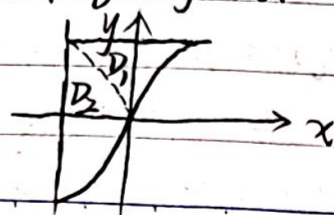
$$1^{\circ} \text{ 若 } f(x,y) = f(-x,-y), \iint_D f(x,y) d\sigma = 2 \iint_{D_1} f(x,y) d\sigma$$

$$2^{\circ} \text{ 若 } f(x,y) = -f(-x,-y), \iint_D f(x,y) d\sigma = 0$$

例1 设平面图形D由曲线 $y = \sin x$, $x = -\frac{\pi}{2}$, $y = 1$ 围成, 则求 $\iint_D (xy^2 - 1) d\sigma$

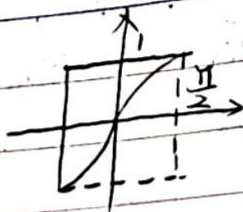
区域D没有包含 $y = -\sin x$.

$$\therefore \iint_D f(x,y) d\sigma = \iint_{D_1} f(x,y) d\sigma + \iint_{D_2} f(x,y) d\sigma \\ = \iint_{D_1} -1 d\sigma + \iint_{D_2} -1 d\sigma = -\iint_D d\sigma$$



$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 - \sin t \, dt = - \left. t + \cos t \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -(\pi - \pi)$$

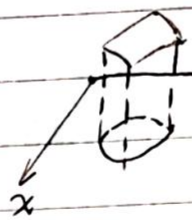
补全后1/2。



△ (2) 轮换对称性 (直角坐标系下)

$$1) \iint_{D: \frac{x^2}{4} + \frac{y^2}{9} \leq 1} (2x^2 + 3y^2) \, dx \, dy = \iint_{D: \frac{x^2}{4} + \frac{y^2}{9} \leq 1} (3x^2 + 2y^2) \, dx \, dy$$

(及内部)



曲面拉体体积不改变
曲线梯形面积亦不变
 $\int_1^2 x^2 \, dx = \int_1^2 t^2 \, dt$

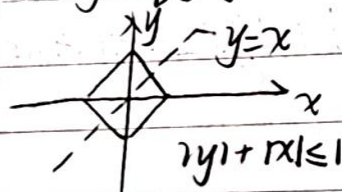
△ 积分值与用何字母表示无关, 同理适用于一重定积分

$$2) \iint_{D: \frac{x^2}{4} + \frac{y^2}{9} \leq 1} (2x^2 + 3y^2) \, dx \, dy = \iint_{D: \frac{x^2}{4} + \frac{y^2}{9} \leq 1} (3x^2 + 2y^2) \, dx \, dy$$

区域D不变 (特殊)

定义: 若将D中x与y对调后D不变, 则有 $\iint_D f(x, y) \, dx \, dy = \iint_D f(y, x) \, dx \, dy$

例1 记 $D = \{(x, y) \mid |x| + |y| \leq 1\}$, 计算下列二重积分。



$$(I) I_1 = \iint_D \frac{af(x) + bf(y)}{f(x) + g(y)} \, dx \, dy$$

若f(x)为定义在R上连续正值函数, $a > 0, b > 0$ 。

$$(II) I_2 = \iint_D (e^{ax} - e^{-ay}) \, dx \, dy, \text{ 常数 } a > 0$$

$$\text{解: } \iint_D \frac{af(x) + bf(y)}{f(x) + f(y)} \, dx \, dy$$

例析 轮换对称性, 适用于 $|y| + |x| \leq 1$

$$(I) I_1 = \iint_D \frac{af(x) + bf(y)}{f(x) + f(y)} \, d\sigma = \iint_D \frac{af(y) + bf(x)}{f(x) + f(y)} \, d\sigma$$

$$\Rightarrow 2I_1 = \iint_D (a+b) \, d\sigma \Rightarrow I_1 = \frac{a+b}{2} \iint_D d\sigma = \frac{a+b}{2} S_D = a+b$$

$$(II) I_2 = \iint_D (e^{ax} - e^{-ay}) \, dx \, dy = \iint_D (e^{ay} - e^{-ax}) \, dx \, dy$$

$$\Rightarrow 2I_2 = \iint_D [(e^{ax} - e^{-ax}) + (e^{ay} - e^{-ay})] \, dx \, dy = 0$$

$$\text{另解: } \iint_D e^{-ay} \, dx \, dy \stackrel{①}{=} \iint_D e^{ay} \, dx \, dy \stackrel{②}{=} \iint_D e^{ax} \, dx \, dy \Rightarrow I_2 = 0$$

例1 设 $f(x)$ 为恒正的连续函数, 证明 $\int_a^b f(x) \, dx \cdot \int_a^b \frac{1}{f(x)} \, dx \geq (b-a)^2$, 其中 $b > a$

例析 证法一: 构成变上限函数, 设 $F(x) = \int_a^x f(t) \, dt \int_a^x \frac{1}{f(t)} \, dt - (x-a)^2$

$$\text{则 } F'(x) = f(x) \int_a^x \frac{1}{f(t)} \, dt + \frac{1}{f(x)} \int_a^x f(t) \, dt - 2(x-a)$$

$$= \int_a^x \left(\frac{f(x)}{f(t)} + \frac{f(t)}{f(x)} \right) \, dt - 2(x-a) \geq \int_a^x 2 \, dt - 2(x-a) = 0$$

$\therefore f(x)$ 恒正 $\therefore f(x)/f(t) + f(t)/f(x) \geq 2$, 当且仅当

则 $f(x) \geq 0 \Rightarrow F(x) \uparrow \Rightarrow F(x) \geq F(a) = 0$ #

$\Rightarrow \int_a^b f(x) dx \geq \int_a^b f(a) dx = (b-a)f(a) \geq 0$ # 核心方法!

方法二: 积分值与字母无关, 构造二重积分.

$\int_a^b f(x) dx = \int_a^b f(y) dy$

故左 = $\int_a^b f(x) dx \cdot \int_a^b f(y) dy$

$= \iint_D f(x)/f(y) d\sigma = \iint_D f(y)/f(x) d\sigma$

$= \frac{1}{2} \iint_D \left[\frac{f(x)}{f(y)} + \frac{f(y)}{f(x)} \right] d\sigma$

$\therefore \frac{f(x)}{f(y)} + \frac{f(y)}{f(x)} \geq 2$

$\therefore \text{左} \geq \iint_D d\sigma = S_D = (b-a)^2 = \text{右}$ #

△(记住) [例2] 计算 $\int_{-\infty}^{+\infty} e^{-x^2} dx$.

[解] 显然 $\int_{-\infty}^{+\infty} e^{-x^2} dx = \int_{-\infty}^{+\infty} e^{-y^2} dy = I$

$\therefore I^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy$

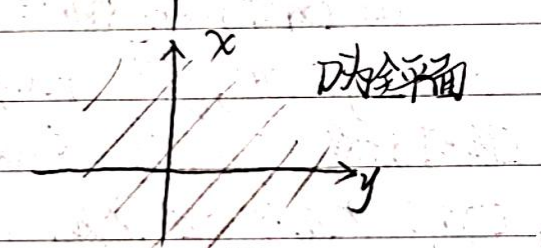
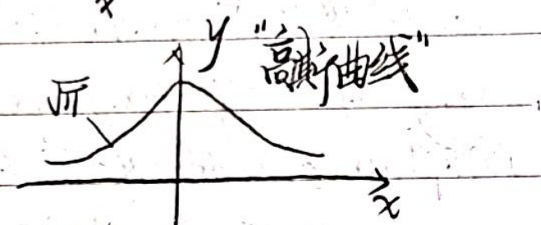
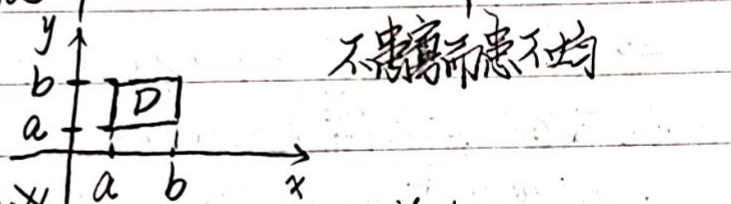
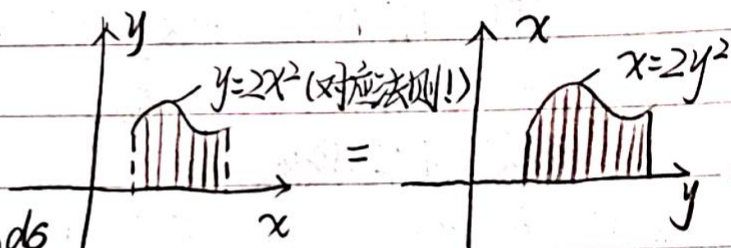
$= \iint_D e^{-(x^2+y^2)} dx dy$

$= \int_0^{2\pi} d\theta \int_0^{+\infty} e^{-r^2} r dr$

$= 2\pi \cdot \frac{1}{2} \int_0^{+\infty} e^{-r^2} d(r^2)$

$= 2\pi \cdot (-\frac{1}{2}) \cdot e^{-r^2} \Big|_0^{+\infty} = \pi$

$\Rightarrow I = \pm\sqrt{\pi} \Rightarrow I = \sqrt{\pi}$ #



$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

二. 计算

1. 直角坐标系

$\iint_D f(x,y) d\sigma = \iint_D f(x,y) dx dy, d\sigma > 0$

(1) X型区域(上下) $\int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y) dy$

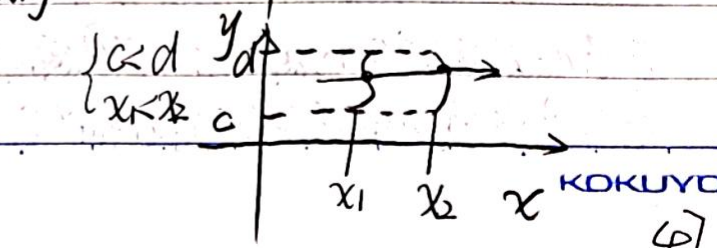
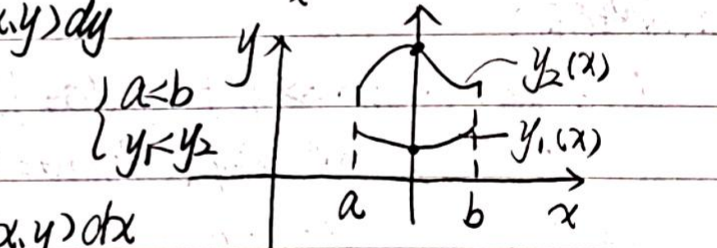
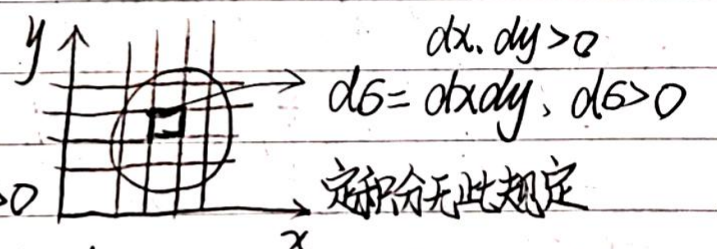
口诀: 后积先定限, 限内画条线

先交写下限, 后交写上限

(2) Y型曲线(左右) $\int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y) dx$

口诀: 后积先定限, 限内画条线

先交左曲线, 右交右曲线



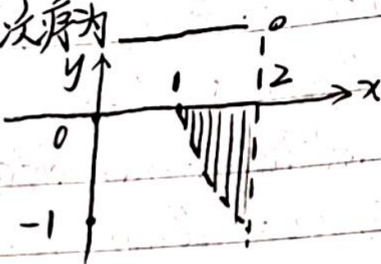
例1 (真题) 交换 $\int_{-1}^0 dy \int_{-y}^{1-y} f(x,y) dx$ 的积分次序为

分析 开始为 y 型. 故 $\int_0^1 dx \int_{-x}^0 f(x,y) dy$

错解: $\int_{-1}^2 dx \int_{1-x}^0 f(x,y) dy$

$dy > 0, \int_{-1}^{1-y} dx, dx < 0$

原题目干非二重积分!



① 检查 $dx \cdot dy > 0$ ② 确认 (修改为) $dx \cdot dy > 0$

$\Rightarrow I = \int_{-1}^0 dy \int_{1-y}^{-y} f(x,y) dx dy = -\int_{-1}^0 dx \int_{1-x}^0 f(x,y) dx dy$ (二重积分)

例2 已知 $I = \int_0^2 dx \int_{\frac{x}{2}}^{\frac{x}{2}} f(x,y) dy + \int_{-2}^{2\sqrt{2}} dx \int_0^{\sqrt{8-x^2}} f(x,y) dy$, 则 $I =$

A. $\int_0^2 dy \int_{\sqrt{8-y^2}}^{\sqrt{8-y^2}} f(x,y) dx$

B. $\int_0^2 dy \int_{\sqrt{8-y^2}}^{\sqrt{8-y^2}} f(x,y) dx$

C. $\int_0^1 dy \int_{\frac{1}{2y}}^{\frac{1}{2y}} f(x,y) dx$

D. $\int_0^2 dy \int_{\frac{1}{2y}}^{\frac{1}{2y}} f(x,y) dx$

分析 (1) (0,2) 在 $y = \frac{x}{2}$ 与 $y=0$

(2) (2, 2√2) 在 $x^2 + y^2 = 8$ 与 $y=0$, 作图 D.

$\therefore I = \int_0^2 dy \int_{\sqrt{8-y^2}}^{\sqrt{8-y^2}} f(x,y) dx$, 选 A.

例3 计算 $\int_{\frac{1}{4}}^{\frac{1}{2}} dy \int_{\frac{1}{2}}^{\frac{1}{2}} e^{\frac{y}{x}} dx + \int_{\frac{1}{2}}^{\frac{1}{4}} dy \int_{\frac{1}{2}}^{\frac{1}{2}} e^{\frac{y}{x}} dx$

分析 “左右型”分块, 出现 $\int e^{\frac{y}{x}} dx, \int e^{-x^2} dx, \int \frac{\cos x}{x} dx, \int \frac{\sin x}{x} dx$ 不易积分.

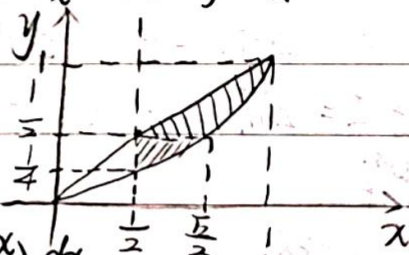
换序 (1) $y \in (\frac{1}{4}, \frac{1}{2}), y = x^2$

(2) $y \in (\frac{1}{2}, 1), y = x$ 与 $y = x^2$.

$\therefore I = \int_{\frac{1}{4}}^{\frac{1}{2}} dx \int_{x^2}^x e^{\frac{y}{x}} dy$

$= \int_{\frac{1}{4}}^{\frac{1}{2}} x e^{\frac{y}{x}} \Big|_{x^2}^x dx = \int_{\frac{1}{4}}^{\frac{1}{2}} x(e - e^x) dx$

$= \frac{3}{8}e - \frac{1}{2}e^{\frac{1}{2}}$



2. 极坐标

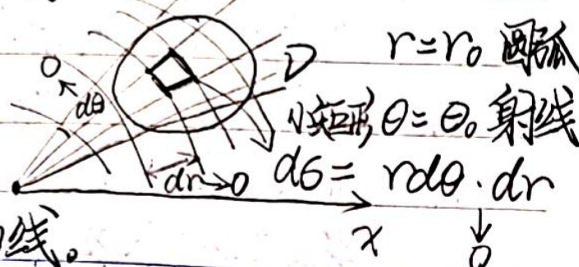
$d\sigma = d\theta \cdot r \cdot dr \Rightarrow \iint_D f(x,y) d\sigma = \int_{\alpha}^{\beta} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr$

$r^2 = x^2 + y^2, \tan \theta = y/x$

$x = r \cos \theta, y = r \sin \theta$

有 $x^2 + y^2$ 及区域与 r 相关, 适合极坐标.

后积先定限, 限内画射线, 先交内曲线, 后交外曲线.



$$I = \iint_D f(x, y) d\sigma = \int_{\alpha}^{\beta} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr$$

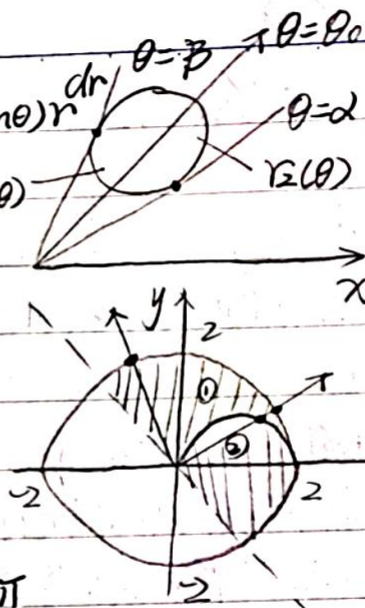
例1 计算 $\iint_D (x^2 + y^2) dx dy$, 其中 D 由 $y = -x$, $x^2 + y^2 = 4$, $y = \sqrt{x - x^2}$ 所围成的在第一象限部分.

分析 $y = \sqrt{x - x^2} \Rightarrow y^2 + (x - 1)^2 = 1, y \geq 0$

1° 当被积函数含 $f(x^2 + y^2)$ 且 D 含圆部分时,

一般选用极坐标系, 若为 $f(x^2 + y^2)$

2° 否则, 则直角坐标系.



$$y = \sqrt{x - x^2} \Rightarrow r = 2 \cos \theta, y = -x \Rightarrow \theta = \frac{3\pi}{4}$$

$$I = \int_0^{\frac{\pi}{2}} d\theta \int_{2 \cos \theta}^2 r^2 r dr + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} d\theta \int_0^2 r^2 r dr$$

$$= \int_0^{\frac{\pi}{2}} 4 - 4 \cos^4 \theta d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 4 d\theta = \frac{9\pi}{4} \quad \#$$

对于②区域 $I = \int_{\frac{\pi}{4}}^0 d\theta \int_0^2 r^3 dr + \int_0^{\frac{\pi}{4}} d\theta \int_0^{2 \cos \theta} r^3 dr \quad \#$

例2 计算 $I = \iint_D \sqrt{1 - r^2} \cos \theta r^2 \sin \theta dr d\theta$

(真) 其中 $D = \{(r, \theta) \mid 0 \leq r \leq \sec \theta, 0 \leq \theta \leq \frac{\pi}{4}\}$.

分析 坐标系错误, 还原

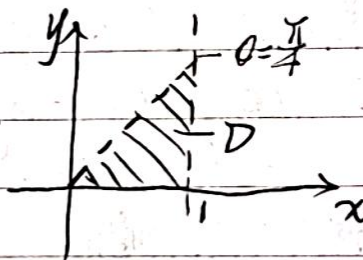
$$r \leq \sec \theta \Rightarrow x \leq 1$$

$$I = \iint_D \sqrt{1 - r^2} (\cos^2 \theta - \sin^2 \theta) \cdot r \sin \theta r dr d\theta$$

$$= \iint_D y \sqrt{1 + y^2 - x^2} dx dy = \int_0^1 dx \int_0^x y \sqrt{1 + y^2 - x^2} dy$$

$$= \frac{1}{3} \int_0^1 (1 - x^2 + y^2)^{\frac{3}{2}} \Big|_{y=0}^{y=x} dx = \frac{1}{3} \int_0^1 1 - (1 - x^2)^{\frac{3}{2}} dx$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} \cos^2 t - \cos^4 t dt = \frac{1}{3} \left(1 - \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \right) = \frac{1}{3} - \frac{\pi}{16} \quad \#$$



第六讲 微分方程

核心考点 1. 概念

2. 一阶方程求解

4/10' 中的一步

3. 高阶方程求解

按步求解, 对号入座

一. 概念

1. $F(x, y, y', \dots, y^{(n)}) = 0$, 含有未知函数 y 及其导数的方程2. 阶数 - 方程中 y 的最高阶导数的阶数如 $y \sin x - y'' = \cos x + 2$ - 二阶 $n=1$ - 一阶, $n \geq 2$ 高阶

3. 通解 - 解中所含独立常数的个数 = 方程的阶数

不定
独立常数 ≠ 任意常数
通解 ≠ 全部解

→ 代入方程, 使之满足等式即为解

二. 一阶方程的求解

1. 变量可分离型

$$dy/dx = f(x, y) \text{ 若 } g(x) \cdot h(y) \Rightarrow \frac{dy}{h(y)} = g(x) dx \Rightarrow \int \frac{dy}{h(y)} = \int g(x) dx$$

[例] 求微分方程 $y' + \frac{1}{y} e^{y^2+3x} = 0$ 的通解[分析] $y e^{-y^2} dy = -e^{3x} dx$, 同时对两边积分

$$-\frac{1}{2} e^{-y^2} = -\frac{1}{3} e^{3x} + C \Rightarrow 2e^{3x} - 3e^{-y^2} = C, \forall C$$

[例] 微分方程的通解 _____ 包含了所有解, 举例说明.

[分析] 不一定, 如求 $(y^2-1)dx = (x-1)y dy$ 的通解

$$\Rightarrow \frac{y dy}{y^2-1} = \frac{dx}{x-1} \Rightarrow \frac{1}{2} \ln |y^2-1| = \ln |x-1| + \frac{1}{2} \ln C_1, \forall C_1 > 0$$

$$\Rightarrow \sqrt{|y^2-1|} = C_1 |x-1| \Rightarrow |y^2-1| = C_1^2 (x-1)^2$$

$$\Rightarrow y^2-1 = \pm C_1^2 (x-1)^2 \text{ 令 } C = C_1^2, \forall C \neq 0$$

故通解为 $y^2-1 = C(x-1)^2, C \neq 0$ *而代入 $y = \pm 1$ 于上式亦成立, 即为其解, 称为奇解.奇解不在通解中. 此题全部解为 $y^2-1 = C(x-1)^2, \forall C$.

考研仅考察

线性系统中通解 = 全部解; 非线性系统中通解 ≠ 全部解.

2. 齐次型方程

$$dy/dx = f\left(\frac{y}{x}\right), \text{ 令 } \frac{y}{x} = u, \text{ 得 } y = u \cdot x \Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot x + u$$

$$\therefore \frac{du}{dx} \cdot x + u = f(u) \Leftrightarrow \int \frac{du}{f(u)-u} = \int \frac{dx}{x} \text{ 回到1}$$

[例] 求解 $(1+e^{-\frac{x}{y}})y dx + (y-x) dy = 0$

[分析] $(1+e^{-\frac{x}{y}})y dx = (x-y) dy$, $\frac{x}{y} = \frac{y}{y}(\frac{x}{y}-1)/(1+e^{-\frac{x}{y}})$
 $\Rightarrow (1+e^{-\frac{x}{y}}) dx = (\frac{x}{y}-1) dy \Rightarrow \frac{dx}{dy} =$

令 $\frac{x}{y} = u \Rightarrow x = uy \Rightarrow dx/dy = du/dy \cdot y + u$, 代入

$$\frac{du}{dy} \cdot y + u = u-1/(1+e^{-u}) \Rightarrow \frac{e^u+1}{e^u+u} du = -\frac{dy}{y}, \text{ 两边同时积分}$$

$$\Rightarrow \ln|e^u+u| = -\ln|y| + \ln C_1, C_1 > 0$$

$$\text{即 } |e^u+u| = \frac{C_1}{|y|} \Rightarrow e^u+u = \pm \frac{C}{y} = cy, C \neq 0$$

$$\text{回代 } e^{\frac{x}{y}} + \frac{x}{y} = cy, \forall C \neq 0 *$$

3. 一阶线性型方程

$$y' + p(x)y = q(x), p(x), q(x) \text{ 已知.}$$

$$(uv)' = u'v + uv' \Rightarrow y' \cdot 1 + y \cdot p(x) \Rightarrow y'e^{\int p(x)dx} + y e^{\int p(x)dx} \cdot p(x)$$

$$\text{故 } e^{\int p(x)dx} y' + e^{\int p(x)dx} y = e^{\int p(x)dx} q(x)$$

$$d[y e^{\int p(x)dx}] / dx = e^{\int p(x)dx} q(x)$$

$$\therefore y e^{\int p(x)dx} = \int e^{\int p(x)dx} q(x) dx$$

$$\Rightarrow y = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} q(x) dx \right] \text{ 公式法}$$

[例] 求 $(y^2+1) dx = y(y-2x) dy$ 的通解.

[分析] $dx/dy = y^2-2xy/(y^2+1) \Rightarrow x' + \frac{2y}{y^2+1} x = \frac{y^2}{y^2+1}$

$$\therefore x = e^{-\int \frac{2y}{y^2+1} dy} \left[\int e^{\int \frac{2y}{y^2+1} dy} \cdot \frac{y^2}{y^2+1} dy \right]$$

$$= \frac{1}{y^2+1} \left(\int y^2 dy \right) = \frac{1}{3(y^2+1)} (y^3 + C), \forall C *$$

三. 二阶方程的求解.

1. $y'' + py' + qy = 0$ p, q 为常数 二阶常系数线性齐次方程

(1) 写 $\lambda^2 + p\lambda + q = 0$, 特征方程 $\Rightarrow \lambda_1, \lambda_2 = \frac{1}{2}(-p \pm \sqrt{p^2 - 4q}) = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$

Δ (2) $\Delta = p^2 - 4q$ $\begin{cases} > 0 \Rightarrow \lambda_1 \neq \lambda_2 \Rightarrow y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ = 0 \Rightarrow \lambda_1 = \lambda_2 \Rightarrow y = (C_1 x + C_2) e^{\lambda x} \end{cases}$

$< 0 \Rightarrow \lambda_1, \lambda_2$ 不存在 $\Rightarrow y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$
 $= \alpha \pm \beta i$

2. $y'' + py' + qy = e^{\alpha x} \cdot P_m(x)$, $P_m(x)$ 为 x 的 m 次多项式

$y_{\text{通解}} = y_{\text{齐次}} + y_{\text{特解}}$

(1) 照搬 $\Rightarrow y_{\text{齐次}}$

(2) 设定 $y^* = e^{\alpha x} Q_m(x) x^k$ $Q_m(x) = \sum_{k=0}^m C_k x^k$, m 次一般多项式

① 一看: 自由项中 α

② 二算: $\lambda^2 + p\lambda + q = 0$, λ_1, λ_2

③ 三比较: 若 $\alpha \neq \lambda_1, \lambda_2$, 则 $k=0$

若 $\alpha = \lambda_1$ 且 $\alpha \neq \lambda_2$, 则 $k=1$

若 $\alpha = \lambda_1 = \lambda_2$, 则 $k=2$

【例】微分方程 $y'' + 2y' + y = \sinh x$ 的特解形式.

【分析】 $\sinh x = \frac{1}{2}(e^x - e^{-x})$ $\cosh x = \frac{1}{2}(e^x + e^{-x})$

1) $y'' + 2y' + y = \frac{1}{2} \cdot e^x$

解 $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1 \Rightarrow y_{\text{齐通}} = (C_1 + C_2 x)e^{-x}$

设 $y_1^* = e^x \cdot x^0 \cdot Q = e^x \cdot A$

2) $y'' + 2y' + y = -\frac{1}{2} \cdot e^{-x}$

设 $y_2^* = e^{-x} \cdot B \cdot x^2$

$\therefore \begin{cases} y_1'' + p y_1' + q y_1 = f_1(x) \\ y_2'' + p y_2' + q y_2 = f_2(x) \end{cases} \Rightarrow (y_1^* + y_2^*)'' + p(y_1^* + y_2^*)' + q(y_1^* + y_2^*) = f_1(x) + f_2(x)$

$\therefore (y_1^* + y_2^*)$ 为 $y'' + 2y' + y = \sinh x$ 的特解 即 $Ae^x + Bx^2e^{-x}$ #

【例】求 $y'' + 5y' + 6y = 2e^{-x}$ 的通解.

【分析】 $y'' + 5y' + 6y = 0 \Rightarrow$ 令 $\lambda^2 + 5\lambda + 6 = 0$ 得 $\lambda_1 = -2, \lambda_2 = -3$

$\therefore y_{\text{齐通}} = C_1 e^{-2x} + C_2 e^{-3x}$

设 $y^* = e^{-x} \cdot A$, $(y^*)' = -Ae^{-x}$, $(y^*)'' = Ae^{-x}$, 代入

$7Ae^{-x} = 2e^{-x} \Rightarrow A = \frac{2}{7} \Rightarrow y^* = \frac{2}{7}e^{-x}$

$\therefore y_{\text{通}} = \frac{2}{7}e^{-x} + C_1 e^{-2x} + C_2 e^{-3x}$ *

其中 C_1, C_2 为任意常数

四. 应用

【例】已知曲线 $y=y(x)$ 经过点 $(1, e^{-1})$, 且在 (x, y) 处切线在 y 轴上截距为 xy , 求曲线方程。

【分析】易知曲线过 (x, y) $(0, xy)$

$$\text{故 } dy/dx = y \cdot \frac{1-x}{x} \Leftrightarrow \int \frac{dy}{y} = \int \frac{1}{x} - 1 dx$$

$$\ln|y| = [\ln|x| - x] + \ln C_1, C_1 > 0$$

$$\text{即 } y = Cx \cdot e^{-x}, C \neq 0$$

$$\text{代入 } (1, e^{-1}), \text{ 得 } C=1 \therefore y = x \cdot e^{-x} \quad \#$$